④ $\int \underset{u}{x^2} \underset{v'}{\cos(x)}\, dx = x^2 \sin(x) - \int \underbrace{2x}_{u'} \underbrace{\sin(x)}_{v} dx$

$= x^2\sin(x) - 2x(-\cos x) + \int 2(-\cos x)\, dx = x^2 \sin(x) + 2x\cos(x) - 2\sin x + c$

⑤ $\int \underbrace{e^x}_{u} \underbrace{\sin x}_{v'} dx = e^x(-\cos x) - \int \underbrace{e^x}_{u} \underbrace{(-\cos x)}_{v'} dx$

$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$

$2\int e^x \sin x\, dx = e^x(\sin x - \cos x)$ $\qquad \int e^x \sin x\, dx = \frac{1}{2}e^x(\cos x - \sin x) + c$

§7.2 Trig integrals

$\int \sin^m x \cos^n x\, dx$?

tools:

- $\cos^2 x + \sin^2 x = 1$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- sub $u = \sin x \quad \frac{du}{dx} = \cos x$
- sub $u = \cos x \quad \frac{du}{dx} = -\sin x$
- parts.

Examples

- $\int \sin^2 x\, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$

- $\int \sin^3 x\, dx = \int (1-\cos^2 x)\sin x\, dx \qquad$ sub $u = \cos x \quad \frac{du}{dx} = -\sin x$

$= \int (1-u^2)\sin x \frac{1}{-\sin x}\, du = \int u^2 - 1\, du = +\frac{1}{3}u^3 - u + c$

$= \frac{1}{3}\cos^3 x - \cos x + c$

moral: squares – double angles

odd powers – do sub $u =$ other trig function

Example

$$\int \sin^4 x \cos^3 x \, dx \qquad \text{try} \quad u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$= \int u^4 (1-u^2) \cos x \frac{1}{\cos x} du = \int u^4 - u^6 \, du = \frac{1}{5}u^5 - \frac{1}{7}u^7 + c$$

$$= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c$$

even powers

$$\int \sin^4 x \cos^2 x \, dx \leftarrow$$ get everything in terms of $\sin(x)$ or $\cos(x)$ and then use parts.

$$\int \sin^4 x (1 - \sin^2 x) \, dx = \int \sin^4 x - \sin^6 x \, dx$$

$$\int \sin^6 x \, dx = \int \underset{u}{\sin^5 x} \, \underset{v'}{\sin x} \, dx \qquad u = \sin^5 x, \quad u' = 5 \sin x \cos x \qquad v' = \sin x, \quad v = -\cos x$$

$$= uv - \int u'v \, dx = -\sin^5 \cos x + \int 5 \sin^4 x \cos^2 x \, dx$$

$$= -\sin^5 x \cos x + 5 \int \sin^4 x (1 - \sin^2 x) dx$$

$$\int \sin^6 x \, dx = -\sin^5 x \cos x + 5 \int \sin^4 x \, dx - 5 \int \sin^6 x \, dx$$

$$6 \int \sin^6 x \, dx = -\sin^5 x \cos x + \underbrace{5 \int \sin^4 x \, dx}$$

do by parts!

other trig functions

recall: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx \qquad u = \cos x, \quad \frac{du}{dx} = -\sin x \qquad = \int \frac{\sin x}{u} \cdot \frac{-1}{\sin x} du$

$$= -\int \frac{1}{u} du = -\ln|u| + c = -\ln|\cos x| + c = \ln|\sec x| + c$$

fact: $\int \sec(x) \, dx = \ln|\sec x + \tan x| + c$ check!

$$\int \csc x \, dx = \ln|\csc x + \cot x| + c$$

other trig function powers

use
- $\cos^2 x + \sin^2 x = 1 \leftrightarrow 1+\tan^2 x = \sec^2 x$
- $u = \sec x$ $\quad \frac{du}{dx} = \sec x \tan x$
- $u = \tan x$ $\quad \frac{du}{dx} = \sec^2 x$
- parts

$$\int \tan^a x \sec^b x \, dx$$

a odd

$$\int \tan^3 x \sec^2 x \, dx = \int \tan^2 x \sec x \,(\tan x . \sec x)\, dx$$

$$= \int (1-\sec^2 x) \sec x (\tan x \sec x)\, dx \qquad u = \sec x \quad \frac{du}{dx} = \sec x \tan x$$

$$= \int (1-u^2)\, u \, du = \frac{1}{2}u^2 - \frac{1}{4}u^4 + c = \frac{1}{2}\sec^2 x - \frac{1}{4}\sec^4 x + c$$

b even $\int \tan^3 x \sec^2 x \, dx \qquad u = \tan x \quad \frac{du}{dx} = \sec^2 x$

$$\int u^3 . du = \frac{1}{4}u^4 + c = \frac{1}{4}\tan^4 x + c$$

a even, b odd : write as powers of sec(x) and integrate by parts.

Example $\int \sin(3x)\cos(2x)\, dx$ useful fact:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ (1)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ (2)

(1)+(2) : $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$$\int \sin(3x)\cos(2x)\, dx = \frac{1}{2}\int \sin(5x) + \sin(x)\, dx = -\frac{1}{10}\cos(5x) - \cos(x) + c$$

Example $\int \cos(4x)\cos(7x)\, dx$ use:

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

so $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$$= \frac{1}{2}\int \cos(11x) + \cos(-3x)\, dx$$

$$= \frac{1}{22}\sin(11x) + \frac{1}{6}\sin(3x) + c$$

§7.3 Trig substitutions

aim: deal with $\sqrt{a^2 - x^2}$ (1) $\quad \sqrt{a^2 + x^2}$ (2) $\quad \sqrt{x^2 - a^2}$ (3)