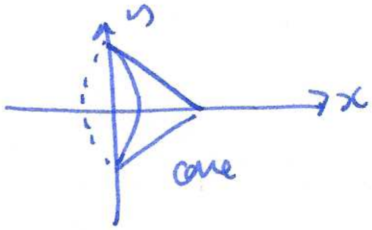
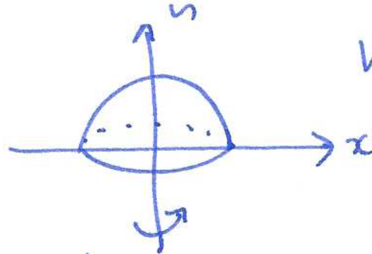
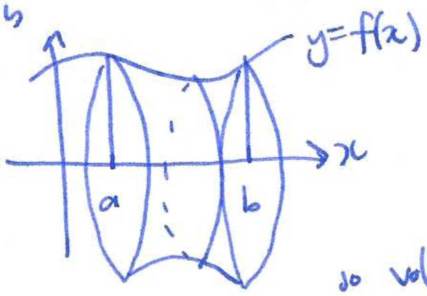
§6.3 Volumes of revolution

Examples





hemisphere



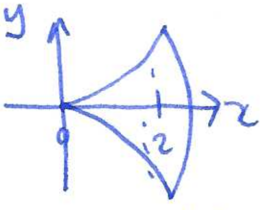
recall : $V = \int_a^b A(x)\,dx$

$\uparrow$ area of vertical cross-section

$= \pi r^2 = \pi y^2 = \pi (f(x))^2$

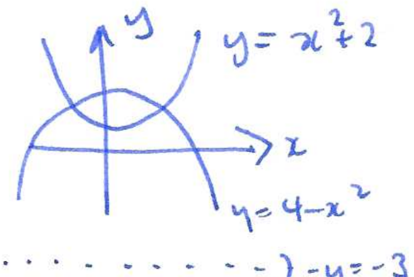
so volume $= V = \int_a^b \pi (f(x))^2\,dx$

Example rotate $y = x^2$ about x-axis



$\pi \int_0^2 (x^2)^2\,dx = \left[\pi\left(\frac{1}{5}x^5\right)\right]_0^2 = \frac{32\pi}{5}$

Example rotate region between $f(x) = x^2+2$, $f(x) = 4-x^2$ about $y=-3$



$V = \int_a^b A(x)\,dx$

a, b : find intersections

$x^2+2 = 4-x^2$

$2x^2 = 2 \quad x = \pm 1$

$A(x)$ = area of cross-section

$A(x) = \pi r_2^2 - \pi r_1^2$

$A(x) = \pi\left[(4-x^2+3)^2 - (x^2+5)^2\right]$

$(7-x^2)^2 \quad (x^2+5)^2$

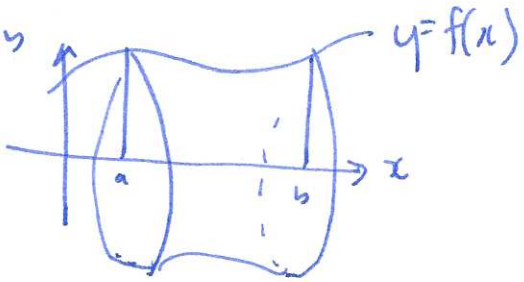
$49 - 14x^2 + x^4 - (x^4 + 10x^2 - 25)$

$24 - 4x^2$

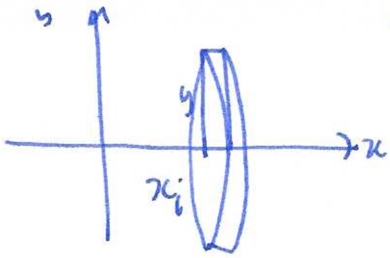
$\int_{-1}^{1} \pi (24 - 4x^2)\,dx$

$= \pi\left[24x - \frac{4x^3}{3}\right]_{-1}^{1} = \left(48 - \frac{8}{3}\right)\pi$

Surface area



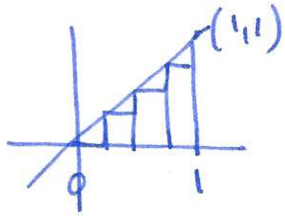
wrong way



area of cylinder $= 2\pi y_i \Delta x_i$

surface area $= \sum 2\pi y_i \Delta x_i = 2\pi \int_a^b y\,dx$

analogy : arc length :

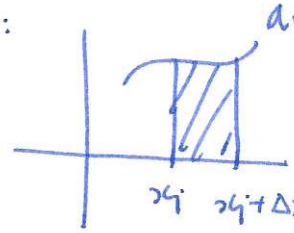


arc length $\neq \sum \Delta x_i$

$\neq \sum \Delta x_i + \Delta y_i$

$\simeq \sum \sqrt{\Delta x_i^2 + \Delta y_i^2}$

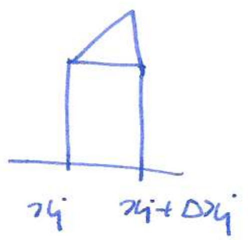
what goes wrong :



area of strip $\approx y(x_i)\Delta x_i$

$\frac{\text{error}}{\text{area}} \to 0$ as $\Delta x_i \to 0$

surface area is like arc length

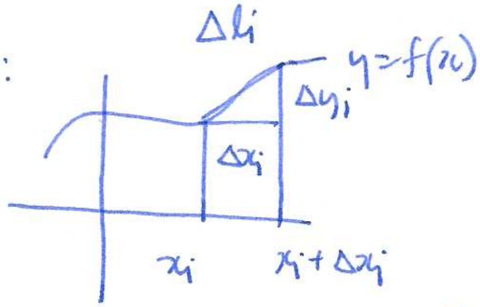


$\frac{\text{error}}{\text{arc length}} \not\to 0$ as $\Delta x_i \to 0$

↑ in fact constant

similarly $\frac{\text{error}}{\text{surface area}} \not\to 0$ as $\Delta x_i \to 0$

right way :



arc length $\simeq \sum \Delta l_i$

$(\Delta l_i)^2 = (\Delta x_i)^2 + (\Delta y_i)^2$

surface area of strip $\approx 2\pi y_i \Delta l_i$

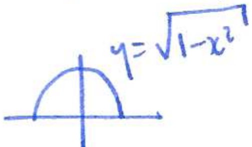
total surface area :

$$\sum 2\pi y_i \Delta l_i = \sum 2\pi\, y_i(x_i) \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$

← doesn't look like an integral

$$= 2\pi \sum y(x_i) \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y_i}\right)^2}\, \Delta x \to$$ surface area $$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

check : surface area of sphere



$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

surface area $= 2\pi \int_{-1}^{1} \sqrt{1-x^2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 4\pi.$