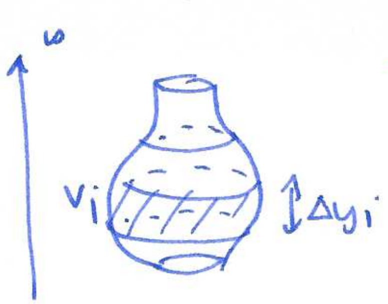
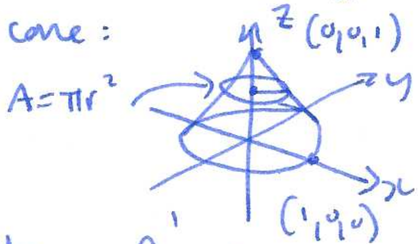


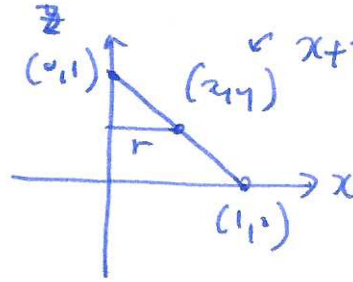
$\leftarrow$ can approximate volume of object by cylinders of height Δy_i, volume

$$V \approx \sum V_i = \sum A(y_i)\,\Delta y_i$$

$$\leadsto \text{ volume } V = \int_a^b A(y)\,dy$$

Example Find volume of cone:



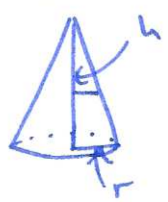


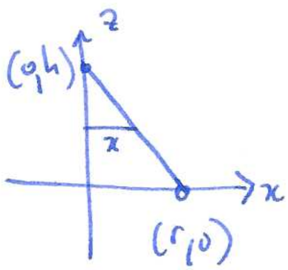
$x + z = 1$, $r = x$

$$V = \int_0^1 A(z)\,dz = \int_0^1 \pi r^2\,dz = \int_0^1 \pi x^2\,dx = \int_0^1 \pi \underset{\overset{\shortparallel}{1-2z+z^2}}{(1-z)^2}\,dz$$

$$= \pi\left[z - z^2 + \tfrac{1}{3}z^3\right]_0^1 = \pi\left(1 - 1 + \tfrac{1}{3}\right) = \frac{\pi}{3}$$

in general:





$$\frac{x}{r} + \frac{z}{h} = 1 \qquad x = r\left(1 - \frac{z}{h}\right)$$

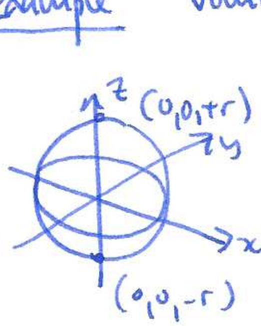
$$V = \int_0^h A(z)\,dz = \int_0^h \pi x^2\,dz$$

$$= \int_0^h \pi r^2\left(1 - \frac{z}{h}\right)^2 dz = \pi r^2 \int_0^h 1 - \frac{2z}{h} + \frac{z^2}{h^2}\,dz = \pi r^2\left[z - \frac{z^2}{h} + \frac{z^3}{3h^2}\right]_0^h$$

$$= \pi r^2\left(h - h + \frac{h}{3}\right) = \frac{1}{3}\pi r^2 h$$

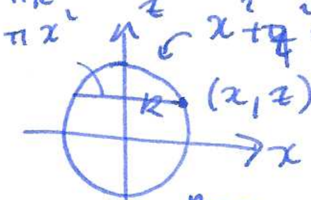
Example Volume of sphere

$x^2 + y^2 + z^2 = r^2$



$$V = \int_{-r}^{r} A(z)\,dz$$

$A = \pi R^2 = \pi x^2$, $x^2 + z^2 = r^2$

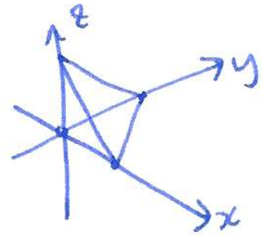


$R = x = \sqrt{r^2 - z^2}$

$$V = \int_{-r}^{r} \pi(r^2 - z^2)\,dz = \pi\left[r^2 z - \tfrac{1}{3}z^3\right]_{-r}^{r}$$

$$= 2\pi\left(r^3 - \tfrac{1}{3}r^3\right) = \tfrac{4}{3}\pi r^3.$$

<u>Example</u> tetrahedron with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$

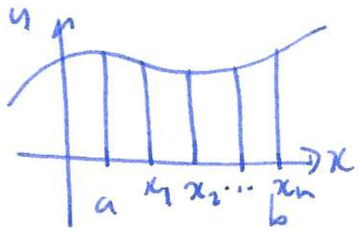


$V = \int_0^1 A(z)\,dz$ area of triangle $= \frac{1}{2}$ base $\times$ height $= \frac{1}{2}(1-z)(1-z)$

$V = \int_0^1 \frac{1}{2}(1-z)^2\,dz = \cdots = \frac{1}{6}$

<u>recall</u>: average value of numbers $a_1, \dots, a_n = \frac{a_1 + \cdots + a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i$

<u>Q</u>: what about average value of function $f(x)$ on an interval $[a,b]$?



average $\approx \frac{1}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right)$.

recall: $\int_a^b f(x)\,dx \approx \left(f(x_1) + \cdots + f(x_n)\right)\Delta x$ $\quad \Delta x = \frac{b-a}{n}$

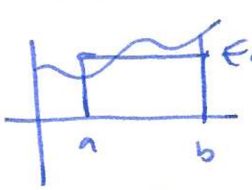
so $\int_a^b f(x)\,dx \simeq$ average $f(x) \cdot (b-a)$ so average $= \frac{1}{b-a}\int_a^b f(x)\,dx$

<u>Example</u> find average value of $\sin(x)$ on $[0,\pi]$:

$$\frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{1}{\pi}\left[-\cos(x)\right]_0^{\pi} = \frac{1}{\pi}(-(-1)+1) = \frac{2}{\pi}$$

<u>useful fact</u> (mean value theorem of integrals)

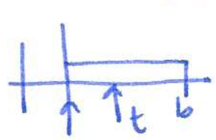
If $f(x)$ is cts on $[a,b]$, then there is a $c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$



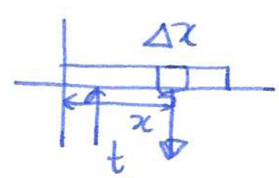
<u>Example</u> find average value of $\frac{\sin(\pi/x)}{x^2}$ on $[1,2]$

<u>Q</u>: which has bigger average on $[0,\pi]$, $\sin(x)$ or $\sin^2 x$?

<u>Application</u>: center of mass



density $\rho(x)$



turning force $(x-t)\Delta x\,\rho(x)$

turning force at t:

$$\sum_i \rho(x)(x_i - t)\Delta x_i \rightsquigarrow \int_a^b \rho(x)(x-t)\,dt = \int_a^b x\,\rho(x)\,dx - t\int_a^b \rho(x)\,dx, \quad t = \frac{\int_a^b x\,\rho(x)\,dx}{\int_0^b \rho(x)\,dx}.$$