Math 232 Calculus 2 Fall 21 Midterm 2b

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (10 points) Find $\int \sin 5x \cos 2x \, dx$.

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ (1)

$\sin(A-B) = \sin A \cos B - \cos A \sin B$ (2)

(1)+(2): $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B$.

$= \frac{1}{2}\int \sin 7x + \sin 3x \, dx = -\frac{1}{14}\cos 7x - \frac{1}{6}\cos 3x + c$

(2) (10 points) Find $\int \cos^3 x \, dx$. $= \int \cos x \cdot \cos^2 x \, dx = \int \cos x \cdot (1 - \sin^2 x) \, dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$.

$\int \cos x \, (1 - u^2) \frac{dx}{du} \, du = \int \cos x \, (1 - u^2) \frac{1}{\cos x} \, du$

$= \int 1 - u^2 \, du = u - \frac{1}{3} u^3 + c = \sin x - \frac{1}{3} \sin^3 x + c$

$\sin^2 x + \cos^2 x = 1$

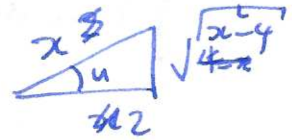
$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x = \sec^2 x - 1.$

(3) (10 points) Find $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$.

$x^2 = 4\sec^2 u \qquad x = 2\sec u \qquad \frac{dx}{du} = 2\sec u \tan u$

$$\int \frac{1}{4\sec^2 u\sqrt{4\sec^2 u - 4}}\frac{dx}{du}\,du = \int \frac{1}{4\sec^2 u \; 2\tan u} 2\sec u \tan u \, du$$



$$= \frac{1}{4}\int \frac{1}{\sec u}\,du = \frac{1}{4}\int \cos u \, du = -\frac{1}{4}\sin u + c$$

$$= -\frac{1}{4}\frac{\sqrt{x^2-4}}{x} + c$$

(4) (10 points) Find $\int \frac{2x^2+x}{(x-2)(x^2+1)}\,dx$.

$$\frac{2x^2+x}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x-2)}{(x-2)(x^2+1)}.$$

try : $x=2$: $10 = 5A \Rightarrow A=2$

$x=0$: $0 = A-2C \Rightarrow C=1$

$x=1$: $3 = 2A+(B+C)(-1)$

$3 = 4-B-1 \Rightarrow B=0$

$$\int \frac{2}{x-2}+\frac{1}{x^2+1}\,dx = 2\ln|x-2| + \tan^{-1}(x) + C$$

$\int u v' dx = uv - \int u' v \, dx$

(5) (10 points) Find $\displaystyle\int_0^\infty e^{-x}\sin(x)\,dx$.

$$\int \underbrace{e^{-x}}_{u} \underbrace{\sin x}_{v'} \, dx = -e^{-x}\cos x - \int -e^{-x} \cdot -\cos x \, dx$$

$$\int e^{-x}\sin x \, dx = -e^{-x}\cos x - \int \underbrace{e^{-x}}_{u} \underbrace{\cos x}_{v'} \, dx$$

$$\int e^{-x}\sin x \, dx = -e^{-x}\cos x - e^{-x}\sin x + \int -e^{-x}\sin x \, dx$$

$$2\int e^{-x}\sin x \, dx = -e^{-x}(\cancel{\cos}\ \sin x + \cos x) + c.$$

$$\lim_{R\to\infty} \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^R = \lim_{R\to\infty} -\frac{1}{2} \underbrace{e^{-R}}_{\to 0} \underbrace{(\sin R + \cos R)}_{1 \cdot 1 \le 2}.$$

$$+\frac{1}{2}$$

$$= \frac{1}{2}.$$

(6) (10 points) Find the degree three Taylor polynomial for $f(x) = \sin(e^x)$ centered at $x = 0$.

$f(x) = \sin(e^x)$ $\qquad f(0) = \sin(1)$

$f'(x) = \cos(e^x).e^x$ $\qquad f'(0) = \cos(1)$

$f''(x) = -\sin(e^x).e^{2x} + \cos(e^x)e^x$ $\qquad f''(0) = -\sin(1) + \cos(1)$

$f^{(3)}(x) = -\cos(e^x).e^{3x} - \sin(e^x).2e^{2x} - \overset{\sin}{\cancel{\cos}}(e^x).e^{2x} + \cos(x)e^x$

$f^{(3)}(0) = -3\sin(1).$

$T_3(x) = \sin(1) + \cos(1)x + (\cos(1) - \sin(1))\frac{x^2}{2!} - 3\sin(1)\frac{x^3}{3!}$

(7) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge or diverge? If it converges, find the exact value.

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1} = \frac{A(n-1)+B(n+1)}{(n+1)(n-1)}$$

$n=1$: $1 = 2B$ $\quad B = 1/2$

$n=-1$: $1 = -2A$ $\quad A = -1/2$

$$\sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}$$

consider partial sums:

$$S_N = \frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{N-2}} - \frac{1}{N} + \cancel{\frac{1}{N-1}} - \frac{1}{N+1}$$

$$= \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}$$

$$\lim_{N\to\infty} \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1} = \frac{3}{2}.$$

(8) Does the series $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^3-1}$ converge or diverge?

try: limit comparison test with $\frac{1}{n^{5/2}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^3-1} \cdot n^{5/2} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = \lim_{n\to\infty} \frac{1}{1-1/n^3} = 1.$$

$\sum_{n=2}^{\infty} \frac{1}{n^{5/2}}$ converges (p-series w/ $p>1$).

$\Rightarrow \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^3-1}$ converges.

(9) Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$ converge or diverge?

limit comparison test w/ $\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+2/n} = 1$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series w/ $p < 1$).

$\Rightarrow$ $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$ diverges.

(10) Find the power series for $f(x) = x\cos(x^2)$. What is the radius of convergence?

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} \cdots + (-1)^n \frac{x^{4n}}{(2n)!}$$

$$x\cos(x^2) = x - \frac{x^5}{2!} + \frac{x^9}{4!} \cdots + (-1)^n \frac{x^{4n+1}}{(2n)!}$$

ratio test: $\lim\limits_{n\to\infty} \left| \frac{(-1)^{n+1} x^{4n+5}}{(2n+2)!} \frac{(-1)^n (2n)!}{x^{4n+1}} \right|$

$= \lim\limits_{n\to\infty} |x|^4 \frac{(2n)!}{(2n+2)!} = \lim\limits_{n\to\infty} |x|^4 \frac{1}{(2n+1)(2n+2)} = 0 \Rightarrow R = \infty.$