Math 232 Calculus 2 Fall 21 Midterm 2a

Name: solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a 3 × 5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 2	
Overall	

(1) (10 points) Find $\int \sin 5x \sin 4x \, dx$.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ ①
$\cos(A-B) = \cos A \cos B + \sin A \sin B$ ②

② − ① $\cos(A-B) - \cos(A+B) = 2 \sin A \sin B$.

$$\int \frac{1}{2}\cos x - \frac{1}{2}\cos 9x \, dx = \frac{1}{2}\sin x - \frac{1}{18}\sin 9x + c$$

(2) (10 points) Find $\int \sin^3 x \, dx$. $= \int \sin x \ \sin^2 x \, dx$

$= \int \sin x \, (1-\cos^2 x) \, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\int \sin x \, (1-u^2) \frac{dx}{du} \, du = \int \sin x \, (1-u^2) \frac{1}{-\sin x} \, dx = \int u^2 - 1 \, du$

$= \frac{1}{3}u^3 - u + c = \frac{1}{3}\cos^3 x - \cos x + c$

$\sin^2 u + \cos^2 u = 1$

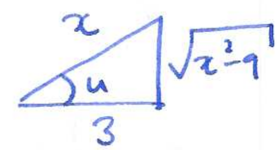
$\tan^2 u + 1 = \sec^2 u$

$\tan^2 u = \sec^2 u - 1.$

(3) (10 points) Find $\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$.

$x^2 = 9\sec^2 u \qquad x = 3\sec u \qquad \frac{dx}{du} = 3\sec u \tan u$

$\int \frac{1}{9\sec^2 u \sqrt{9\sec^2 u - 9}} \frac{dx}{du}\,du = \int \frac{1}{9\sec^2 u \; 3\tan u} 3\sec u \tan u \,du$



$= \frac{1}{9}\int \frac{1}{\sec u}\,du = \frac{1}{9}\int \cos u\,du = -\frac{1}{9}\sin u + c$

$= -\frac{1}{9}\frac{\sqrt{x^2-9}}{x} + c$

(4) (10 points) Find $\displaystyle\int \frac{2x^2 - x}{(x+2)(x^2+1)}\,dx$.

$$\frac{2x^2-x}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x+2)}{(x+2)(x^2+1)}.$$

try: $x=-2$: $10 = 5A \Rightarrow A = 2$

$x = 0$: $0 = A + 2C \Rightarrow C = -1$

$x = 1$: $1 = 2A + (B+C).3$

$1 = 4 + 3B - 3 \Rightarrow B = 0$

$$\int \frac{2}{x+2} - \frac{1}{x^2+1}\,dx = 2\ln|x+2| - \tan^{-1}(x) + C$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(5) (10 points) Find $\int_0^\infty e^{-x}\cos(x)\,dx$.

$$= \lim_{R\to\infty} \int_0^R \underbrace{e^{-x}}_{u} \underbrace{\cos x}_{v'} dx = \lim_{R\to\infty} \left[e^{-x}\sin x \right]_0^R - \int_0^R -e^{-x}\sin x\,dx$$

$$= \lim_{R\to\infty} \underbrace{e^{-R}}_{\to 0} \cdot \underbrace{\sin R}_{1 \cdot 1 \le 1} + \int_0^R \underbrace{e^{x}}_{u} \underbrace{\sin x}_{v'}\,dx$$

$$\lim_{R\to\infty} \int_0^R e^{-x}\cos x = \lim_{R\to\infty} \left[e^{-x} \cdot -\cos x \right]_0^R - \int_0^R -e^{-x} \cdot -\cos x\,dx$$

$$\lim_{R\to\infty} \int_0^R e^{-x}\cos x = \lim_{R\to\infty} -\underbrace{e^{-R}}_{\to 0} \underbrace{\cos R}_{\le 1} + 1 - \int_0^R e^{-x}\cos x\,dx$$

$$\lim_{R\to\infty} 2\int_0^R e^{-x}\cos x\,dx = 1$$

$$\int_0^\infty e^{-x}\cos x\,dx = \frac{1}{2}$$

(6) (10 points) Find the degree three Taylor polynomial for $f(x) = \cos(e^x)$ centered at $x = 0$.

$f(x) = \cos(e^x)$ $\qquad$ $f(0) = \cos(1)$

$f'(x) = -\sin(e^x).e^x$ $\qquad$ $f'(0) = -\sin(1)$

$f''(x) = -\cos(e^x).e^{2x} - \sin(e^x).e^x$ $\qquad$ $f''(0) = -\cos(1) - \sin(1).$

$f^{(3)}(x) = +\sin(e^x).e^{3x} - \cos(e^x).2e^{2x} - \cos(e^x).e^{2x} - \sin(e^x).e^x.$

$f^{(3)}(0) = -3\cos(1)$

$T_3(x) = \cos(1) - \sin(1)x - (\cos(1)+\sin(1))\frac{x^2}{2!} - 3\cos(1)\frac{x^3}{3!}$

(7) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge or diverge? If it converges, find the exact value.

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1} = \frac{A(n-1)+B(n+1)}{(n+1)(n-1)}$$

$n=1$: $1 = 2B$ $\quad B = 1/2$

$n=-1$: $1 = -2A$ $\quad A = -1/2$

$$\sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}$$ consider partial sums:

$$S_N = \frac{1}{1} - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cdots + \cancel{\frac{1}{N-2}} - \frac{1}{N} + \cancel{\frac{1}{N-1}} - \frac{1}{N+1}$$

$$= \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1}$$

$$\lim_{N\to\infty} \frac{3}{2} - \frac{1}{N} - \frac{1}{N+1} = \frac{3}{2}.$$

(8) Does the series $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^3-1}$ converge or diverge?

try: limit comparison test with $\frac{1}{n^{5/2}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^3-1} \cdot n^{5/2} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = \lim_{n\to\infty} \frac{1}{1-1/n^3} = 1.$$

$\sum_{n=2}^{\infty} \frac{1}{n^{5/2}}$ converges (p-series w/ $p>1$).

$\Rightarrow \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^3-1}$ converges.

(9) Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$ converge or diverge?

limit comparison test w/ $\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+2} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+2/n} = 1$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series w/ $p<1$).

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+2}$ diverges.

(10) Find the power series for $f(x) = x\sin(x^2)$. What is the radius of convergence?

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} \cdots\cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$x\sin(x^2) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} \cdots - + (-1)^n \frac{x^{4n+3}}{(2n+1)!} + \cdots$$

ratio test:

$$\lim_{n\to\infty} \left| (-1)^{n+1} \frac{x^{4n+7}}{(2n+3)!} \frac{(-1)^n (2n+1)!}{x^{4n+3}} \right|.$$

$$= \lim_{n\to\infty} |x|^4 \frac{(2n+1)!}{(2n+3)!} = |x|^4 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = 0$$

$\Rightarrow R = \infty.$