MTH 232 SMT2 Solutions

Q1 $\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)\,dx$ $\quad u=\cos x \quad \frac{du}{dx} = -\sin x$ $\quad \int \sin x\,(1-u^2)\frac{dx}{du}\,du$

$= \int \sin x\,(1-u^2)\frac{1}{-\sin x}\,du = \int u^2 - 1\,du = \frac{1}{3}u^3 - u + c = \frac{1}{3}\cos^3 x - \cos x + c$

Q2 $\int \cos 8x \cos 3x\,dx$

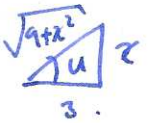
$\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\Big\}\ \cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$= \frac{1}{2}\int \cos 11x + \cos 5x\,dx = \frac{1}{22}\sin 11x + \frac{1}{10}\sin 5x + c$

Q3 $\int \frac{x}{\sqrt{x^2+9}}\,dx$

$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$

$\overset{x}{\cancel{t}} = 3\tan \overset{u}{\cancel{x}}$ $\quad \frac{dx}{du}\cancel{\frac{du}{dx}} = 3\sec^2 \cancel{x}\,u$ $\quad \int \frac{x}{\sqrt{9\tan^2 u + 9}}\frac{dx}{du}\,du$



$\int \frac{3\tan u}{3\sqrt{\sec^2 u}}\,3\sec^2 u\,du = 3\int \tan u \sec u\,du = 3\sec u + c$

$= 3\frac{\sqrt{9+x^2}}{3} = \sqrt{9+x^2} + c$

Q4 $\int \frac{x^2+5}{(x+1)^2(x-2)}\,dx$ $\quad \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{c}{x-2} = \frac{x^2+5}{(x+1)^2(x-2)}$

$A(x+1)(x-2) + B(x-2) + c(x+1)^2 = x^2+5$

$x=-1: \quad -3B = 6 \quad B = -2$
$x=2: \quad 9c = 9 \quad c = 1$
$x=0: \quad -2A - 2B + c = 5$
$\quad -2A + 4 + 1 = 5 \quad A = 0$

$\int \frac{-2}{(x+1)^2} + \frac{1}{x-2}\,dx = \frac{2}{(x+1)} + \ln|x-2| + c$

Q5 $\int_0^1 x^2 \ln x^3\,dx \overset{\circledast}{} = \lim_{R\to 0}\int_R^1 3x^2 \ln x\,dx$ $\qquad \int uv'\,dx = uv - \int u'v\,dx$

$u = \ln x \quad u' = \frac{1}{x}$
$v' = 3x^2 \quad v = x^3$

$\lim_{R\to 0}\left[x^3 \ln x\right]_R^1 - \int_R^1 \frac{1}{x}\cdot x^3\,dx$

$= \lim_{R\to 0} -R^3 \ln R - \left[\frac{1}{3}x^3\right]_R^1 = \lim_{R\to 0} -R^3\ln R - \frac{1}{3} + \frac{1}{3}R^3$

$\lim_{R\to 0}\frac{\ln R}{-R^{-3}} \overset{l'H}{=} \lim_{R\to 0}\frac{1/R}{3R^{-4}} = \lim_{R\to 0}\frac{1}{3}R^3 = 0$ $\quad$ so $\circledast = -\frac{1}{3}$.

Q6 $\begin{array}{r} 1 \\ x-1\overline{)\,x} \\ \underline{x-1} \\ 1 \end{array}$ $\int_0^2 \frac{x}{x-1}\,dx \overset{}{=} \circledast$ $\int_0^2 1 + \frac{1}{x-1}\,dx = \lim_{R\to 1} \int_0^R 1+\frac{1}{x-1}\,dx + \lim_{R\to 1}\int_R^2 1+\frac{1}{x-1}\,dx$

$= \lim_{R\to 1} \Big[x + \ln|x-1| \Big]_0^R = \lim_{R\to 1} R + \ln|R-1| = -\infty$ so $\circledast$ DNE.

Q7 $\int_0^\infty \frac{1}{16x^2+1}\,dx$ $\quad 16x^2 = u^2 \quad 4x = u \quad \frac{dx}{du} = \frac{1}{4} \quad \lim_{R\to\infty} \int_0^R \frac{1}{u^2+1} \frac{dx}{du}\,du$

$= \lim_{R\to\infty} \frac{1}{4}\int_0^{R/4} \frac{1}{1+u^2}\,du = \lim_{R\to\infty} \frac{1}{4}\Big[\tan^{-1}(u)\Big]_0^{R/4} = \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{8}$

Q8 $f(x) = x^{1/3}$

$f'(x) = \frac{1}{3}x^{-2/3}$ ← not defined at $x=0$, so no Taylor series centered at 0

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\qquad f(1) = 1 \quad f'(1) = \frac{1}{3} \quad f''(1) = -\frac{2}{9} \quad f^{(3)}(1) = \frac{10}{27}$

$f^{(3)}(x) = \frac{10}{27}x^{-8/3}$ $\qquad T_3(x) = 1 + \frac{1}{3}(x-1) - \frac{2}{9}\frac{(x-1)^2}{2!} + \frac{10}{27}\frac{(x-1)^3}{3!}$

$f^{(4)}(x) = -\frac{80}{81}x^{-11/3}$.

$\sqrt[3]{2} \approx T_3(2) = 1 + \frac{1}{3} - \frac{2}{9.2} + \frac{10}{27.6} =$ ~~$\frac{27+9-6+10}{27} = \frac{40}{27}$~~ ≈ 1.283951

error bound: $\left|\sqrt[3]{2} - \frac{40}{27}\right| \leq \frac{K(2-1)^4}{4!}$ where K is an upper bound for $\left|\frac{-80}{81}x^{-11/3}\right|$

on $[1,2]$ $\left|\frac{-80}{81}x^{-11/3}\right| \leq 1$ so $\left|\sqrt[3]{2} - \frac{40}{27}\right| \leq \frac{1}{4!}$

Q9 $a_n = \frac{3^n}{n!} = \frac{3.3.\ldots\ldots 3}{1.2.3.\ldots\ldots n} = \frac{3}{1}\cdot\frac{3}{2}\cdot\underbrace{\frac{3}{3}\,\frac{3}{4}\cdots\frac{3}{n-1}}_{\leq 1}\cdot\frac{3}{n} \leq \frac{27}{4n} \to 0$ as $n\to\infty$

so $a_n \to 0$ as $n\to\infty$

Q10 $\sum_{n=1}^{\infty} e^{-2n} = e^{-2} + e^{-4} + e^{-6} + \cdots$ geometric series $= \frac{e^{-2}}{1-e^{-2}}$

Q11 $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$ $\qquad \frac{1}{n^2+5n+6} = \frac{1}{(n+2)(n+3)} = \frac{A}{n+2} + \frac{B}{n+3} = \frac{A(n+3)+B(n+2)}{(n+2)(n+3)}$

$n=-2: \ A = 1 \qquad n=-3: \ -B = 1$

$= \sum_{k=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+3} \Rightarrow s_N = \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{N+2} - \frac{1}{N+3} = \frac{1}{3} - \frac{1}{N+3}$

so $\lim_{N\to\infty} s_N = \frac{1}{3}$

Q12 $\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = 1$ so does not converge by divergence test.

Q13 $\sum_{n=1}^{\infty} \frac{\ln(n)^2}{n^3}$ claim: $\ln(n)^2 < n$ for large n

check: $\lim_{n\to\infty} \frac{\ln(n)^2}{n} \overset{l'H}{=} \lim_{n\to\infty} \frac{2\ln(n)\frac{1}{n}}{1}$

$= \lim_{n\to\infty} \frac{2\ln(n)}{n} \overset{l'H}{=} \lim_{n\to\infty} \frac{2/n}{1} = \lim_{n\to\infty} \frac{2}{n} = 0$ ✓.

comparison test: $\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ ← converges as p-series with $p>1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln(n)^2}{n^3}$ converges.

Q14 $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ ratio test: $\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$ $\Rightarrow$ converges.

Q15 $\sum_{n=1}^{\infty} \frac{n\cos(n)}{n^4+1}$ consider absolute value series: $\sum_{n=1}^{\infty} \frac{n|\cos(n)|}{n^4+1}$ comparison test:

$\frac{n|\cos(n)|}{n^4+1} \leq \frac{n}{n^4+1}$ limit comparison test with $\frac{1}{n^3}$: $\lim_{n\to\infty} \frac{1/n^3}{n/n^4+1}$

$= \lim_{n\to\infty} \frac{n^4+1}{n^4} = \lim_{n\to\infty} 1 + \frac{1}{n^4} = 1 \Rightarrow$ converges as $\sum \frac{1}{n^3}$ converges as p-series w/ $p>1$.

Q16 $\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}$ limit comparison test w/ $\frac{1}{n}$ $\lim_{n\to\infty} \frac{1/n}{n^3/n^4+1} = \lim_{n\to\infty} \frac{n^4+1}{n^4} = \lim_{n\to\infty} 1 + \frac{1}{n^4} = 1$

diverges as $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges as p-series w/ $p \leq 1$.

Q17 $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ ratio test: $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^3} \frac{n^3}{x^n} \right| = \lim_{n\to\infty} |x| \frac{n^3}{(n+1)^3}$

$= \lim_{n\to\infty} |x| \frac{1}{(1+1/n)^3} = |x|$ so converges for $|x| < 1$

(converges for $|x| \leq 1$: $x = -1$: alternating series test)
$x = 1$: p-series w/ $p > 1$

Q18 $f(x) = \sin(\sqrt{x})$.

$f'(x) = \cos(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$

$f''(x) = -\sin(\sqrt{x}) \cdot \frac{1}{4}x^{-1} + \cos(\sqrt{x}) \cdot -\frac{1}{4}x^{-3/2}$

$f(1) = \sin(1)$

$f'(1) = \frac{1}{2}\cos(1)$

$f''(1) = -\frac{1}{4}\sin(1) - \frac{1}{4}\cos(1)$

$\sin(1) + \frac{1}{2}\cos(1)(x-1) + \left(-\frac{1}{4}\sin(1) - \frac{1}{4}\cos(1)\right)\frac{(x-1)^2}{2!}$

Q19 $x^2 e^{-x^3} = x^3\left(1 - x^3 + \frac{x^6}{2!} + \cdots\right)$

$= x^3 - x^6 + \frac{x^9}{4} + \cdots$