MTH 232 SMT1 Solutions

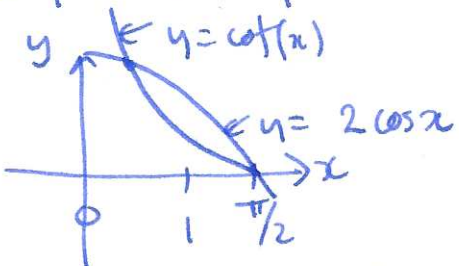
Q1 $\displaystyle\int \frac{\sin x}{1-\cos x}\,dx$ try: $u = \cos x$, $\dfrac{du}{dx} = -\sin x$ $\displaystyle\int \frac{\sin x}{1-u}\frac{dx}{du}\,du =$

$$\int \frac{\sin x}{1-u}\,\frac{1}{-\sin x}\,du = \int \frac{1}{u-1}\,du = \ln|u-1| + c = \ln|\cos x - 1| + c$$

Q2 $$\int \frac{\sin x}{1-\cos^2 x}\,dx = \int \frac{\sin x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\,dx = \int \sec x\,dx$$

$$= \ln|\sec x + \tan x| + c$$

Q3

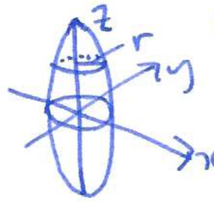


intersection: $\cot(x) = 2\cos x = \dfrac{\cos x}{\sin x} \iff \sin x = \dfrac{1}{2}$, $x = \pi/6$.

$$\int_{1/4}^{\pi/6} \cot(x) - 2\cos x\,dx + \int_{\pi/6}^{1} 2\cos x - \cot(x)\,dx$$

$$= \Big[\ln|\sin x| - 2\sin x\Big]_{1/4}^{\pi/6} + \Big[2\sin x - \ln|\sin x|\Big]_{\pi/6}^{1} \approx 0.36\ldots$$

Q4



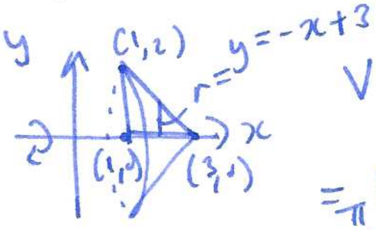
a) $A(z) = \pi r^2$ $r^2 = x^2 + y^2 = \dfrac{16 - z^2}{3}$.

b) $$\int_{-4}^{4} \pi\left(\frac{16-z^2}{3}\right)dz = \frac{\pi}{3}\left[16z - \frac{1}{3}z^3\right]_{-4}^{4} = \frac{\pi}{3}\left[\left(4^3 - \frac{1}{3}4^3\right) - \left(-4^3 + \frac{1}{3}4^3\right)\right]$$

$$= \frac{\pi\, 4^4}{9}$$

Q5 $$\frac{1}{4}\int_0^4 e^{-2x}\,dx = \frac{1}{4}\left[-\frac{1}{2}e^{-2x}\right]_0^4 = \frac{1}{4}\left(-\frac{1}{2}e^{-8} + \frac{1}{2}\right) = \frac{1}{8}\left(1 - e^{-8}\right).$$

Q6

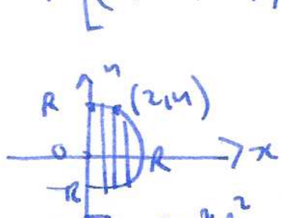


$$V = \int_1^3 \pi r^2\,dx = \int_1^3 \pi(3-x)^2\,dx$$

$$= \pi\int_1^3 9 - 6x + x^2\,dx = \pi\left[9x - 3x^2 + \frac{1}{3}x^3\right]_1^3.$$

$$= \pi\left[(27 - 27 + 9) - \left(9 - 3 + \frac{1}{3}\right)\right] = \frac{8\pi}{3}$$

Q7



$r = x$

$h = 2y = 2\sqrt{R^2 - x^2}$

$$V = \int_0^R 2\pi r h\,dx = \int_0^R 2\pi x\, 2\sqrt{R^2 - x^2}\,dx$$

$u = R^2 - x^2$ $\dfrac{du}{dx} = -2x$

$$= \int_{R^2}^{0} 4\pi x\sqrt{u}\,\underbrace{\frac{dx}{du}}_{-\frac{1}{2x}}\,du = \int_0^R 2\pi u^{1/2}\,du$$

$$= 2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{R^2} = \frac{4\pi R^3}{3}.$$

Q8 $\int x^2\ln(x-1)\,dx.$ $u = x-1$, $\frac{du}{dx} = 1$

$$\int \underbrace{(u+1)^2}_{v'}\underbrace{\ln(u)}_{w}\,du$$

$v' = (u+1)^2 = u^2+2u+1$

$v = \frac{1}{3}u^3+u^2+u$

$w = \ln(u)$ $w' = \frac{1}{u}.$

$$\oint \left(\frac{1}{3}u^3+u^2+u\right)\ln(u) - \int\left(\frac{1}{3}u^3+u^2+u\right)\frac{1}{u}\,du$$

$$= \left(\frac{1}{3}u^3+u^2+u\right)\ln(u) - \int \frac{1}{3}u^2+u+1\,du = \left(\frac{1}{3}u^3+u^2+u\right)\ln(u) - \frac{1}{9}u^3 - \frac{1}{2}u^2 - u + c$$

$$= \frac{1}{3}(x-1)^3 + (x-1)^2 + (x-1)\ln(x-1) - \frac{1}{9}(x-1)^3 - \frac{1}{2}(x-1)^2 - (x-1) + c$$

Q9 $\int \underbrace{e^{-2x}}_{u}\underbrace{\sin(3x)}_{v'}\,dx$

$u = e^{-2x}$ $v' = \sin(3x)$

$u' = -2e^{-2x}$ $v = -\frac{1}{3}\cos(3x)$

$$\int e^{-2x}\sin(3x)\,dx = e^{-2x}\cdot -\frac{1}{3}\cos(3x) - \int -2e^{-2x}\cdot -\frac{1}{3}\cos(3x)\,dx$$

$$\int e^{-2x}\sin(3x)\,dx = -\frac{1}{3}e^{-2x}\cos(3x) - \int \underbrace{\frac{2}{3}e^{-2x}}_{u}\underbrace{\cos(3x)}_{v'}\,dx$$

$u = \frac{2}{3}e^{-2x}$ $v' = \cos(3x)$

$u' = -\frac{4}{3}e^{-2x}$ $v = \frac{1}{3}\sin(3x)$

$$\int e^{-2x}\sin(3x)\,dx = -\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{9}e^{-2x}\sin(3x) + \int -\frac{4}{3}e^{-2x}\cdot\frac{1}{3}\sin(3x)\,dx$$

$$\frac{13}{9}\int e^{-2x}\sin(3x)\,dx = -\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{9}e^{-2x}\sin(3x).$$

$$\int e^{-2x}\sin(3x)\,dx = -\frac{3}{13}e^{-2x}\cos(3x) - \frac{2}{13}e^{-2x}\sin(3x) + c.$$

Q10 $\int \underbrace{x}_{u}\underbrace{e^{-x}\cos(x)}_{v'}\,dx$

$u = x$ $v' = e^{-x}\cos x.$

$u' = 1$

$$v = \int \underset{a}{e^{-x}}\,\underset{b'}{\cos(x)}\,dx = \underset{a}{e^{-x}}\,\underset{b}{\sin x} - \int \underset{a'}{-e^{-x}}\,\underset{b}{\sin x}\,dx$$

$$\int e^{-x}\cos(x)\,dx = e^{-x}\sin x + \int \underset{a}{e^{-x}}\,\underset{b'}{\sin x}\,dx = e^{-x}\sin x + \underset{a}{e^{-x}}\cdot\underset{b}{-\cos x} - \int \underset{a'}{-e^{-x}}\cdot\underset{b}{-\cos x}\,dx$$

$$2\int e^{-x}\cos x\,dx = e^{-x}\sin x - e^{-x}\cos x + c$$

$$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}\sin x - \frac{1}{2}e^{-x}\cos x + c$$

$$\int x e^{-x}\cos x\,dx = x\left(\tfrac{1}{2}e^{-x}\sin x-\tfrac{1}{2}e^{-x}\cos x\right)-\int \tfrac{1}{2}e^{-x}\sin x-\tfrac{1}{2}e^{-x}\cos x\,dx$$

$$= \tfrac{1}{2}xe^{-x}(\sin x-\cos x)-\tfrac{1}{2}\int e^{-x}\sin x\,dx-\tfrac{1}{2}\underbrace{\int e^{-x}\cos x\,dx}_{=\frac{1}{2}e^{-x}(\sin x-\cos x)}$$

$$\int e^{-x}\sin x\,dx = e^{-x}\cdot-\cos x-\int -e^{-x}\cdot-\cos x\,dx = -e^{-x}\cos x-\int e^{-x}\cos x\,dx$$

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x-e^{-x}\sin x+\int -e^{-x}\sin x\,dx$$

$$2\int e^{-x}\sin x\,dx = -e^{-x}(\cos x+\sin x)+c$$

$$\int xe^{-x}\cos x\,dx = \tfrac{1}{2}xe^{-x}(\sin x-\cos x)+\tfrac{1}{4}e^{-x}(\cos x+\sin x)-\tfrac{1}{4}e^{-x}(\sin x-\cos x)+c$$

$$= \tfrac{1}{2}xe^{-x}(\sin x-\cos x)+\tfrac{1}{2}e^{-x}\cos x+c$$

Q11 $\displaystyle\int_0^{\pi/2}\sin^3 x\cos^2 x\,dx$ $\quad u=\cos x \quad \frac{du}{dx}=-\sin x$ $\quad \displaystyle\int_1^0 \sin^2 x\sin x\, u^2\frac{dx}{du}\,du$

$$= \int_1^0 (1-\cos^2 x)\sin x\, u^2\frac{1}{-\sin x}\,du = \int_0^1(1-u^2)u^2\,du = \int_0^1 u^2-u^4\,du = \left[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right]_0^1$$

$$= \tfrac{1}{3}-\tfrac{1}{5} = \tfrac{2}{15}$$

Q12 $\displaystyle\int \underset{A}{\sin(6x)}\,\underset{B}{\cos(4x)}\,dx$

$$\overset{\cos}{\sin}(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos(A-B) = \sin A\cos B+\cos A\sin B$$
$$\cos(A+B)+\cos(A-B) = 2\sin A\cos B$$

$$= \tfrac{1}{2}\int\cos(10x)+\cos(2x)\,dx = \tfrac{1}{20}\sin(10x)+\tfrac{1}{4}\cos(2x)+c$$

Q13 $\displaystyle\int\frac{x^2}{\sqrt{4x^2+1}}\,dx$ $\quad x=\tfrac{1}{2}\tan u \quad \frac{dx}{du}=\tfrac{1}{2}\sec^2 u$ $\quad \displaystyle\int\frac{\frac{1}{4}\tan^2 u}{\sqrt{\tan^2 u+1}}\frac{dx}{du}\,du$

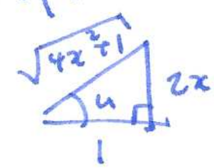
$$= \tfrac{1}{4}\int\frac{\tan^2 u}{\sec u}\tfrac{1}{2}\sec^2 u\,du = \tfrac{1}{8}\int \underset{\sec^2 u-1}{\underset{\shortparallel}{\tan^2 u}}\sec u\,du = \tfrac{1}{8}\int\sec^3 u-\sec u\,du$$

$$\int\sec^3 u\,du = \int\sec^2 u\sec u\,du = \tan u\sec u-\int\tan u\cdot\sec u\tan u\,du$$

$$\int\sec^3 u\,du = \tan u\sec u-\int\underset{\sec^2 u-1}{\underset{\shortparallel}{\tan^2 u}}\sec u\,du = \tan u\sec u-\int\sec^3 u-\sec u\,du$$

$2\int \sec^3 u\,du = \tan u\sec u + \ln|\sec u + \tan u| + c$

$\frac{1}{8}\int \sec^3 u - \sec u\,du = \frac{1}{16}\tan u\sec u + \frac{1}{16}\ln|\sec u+\tan u| - \frac{1}{8}\ln|\sec u+\tan u| + c$



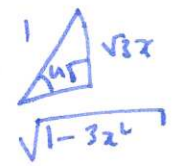
$= \frac{1}{16}\tan u\sec u - \frac{1}{16}\ln|\sec u+\tan u| + c$

$= \frac{1}{16}\, 2x\cdot\frac{1}{\sqrt{4x^2+1}} - \frac{1}{16}\ln\left|\frac{1}{\sqrt{4x^2+1}} + 2x\right| + c$

Q14 $\int\sqrt{x^2-9}\,dx$ $\quad x = 3\sec u$. $\quad \frac{dx}{du} = 3\sec u\tan u$ $\quad \int\sqrt{9\sec^2 u - 9}\,\frac{dx}{du}\,du$

$= 3\int \underbrace{\sqrt{\sec^2 u - 1}}_{\tan^2 u}\, 3\sec u\tan u\,du = 9\int \sec u\tan^2 u\,du$ see previous Q.

Q15 $\int\frac{x}{\sqrt{1-3x^2}}\,dx$ $\quad x = \frac{1}{\sqrt3}\sin u$ $\quad \frac{dx}{du} = \frac{1}{\sqrt3}\cos u$ $\quad = \int\frac{\frac{1}{\sqrt3}\sin u}{\sqrt{1-3\cdot\frac13\sin^2 u}}\,\frac{dx}{du}\,du$



$\frac{1}{\sqrt3}\int\frac{\sin u}{\underbrace{\sqrt{1-\sin^2 u}}_{\cos^2 u}}\,\frac{1}{\sqrt3}\cos u\,du = \frac13\int\frac{\sin u\cos u}{\cos u}\,du = \frac13\int\sin u\,du = -\frac13\cos u + c$

$= -\frac13\sqrt{1-3x^2} + c$

Q16 $\int\tan^3 3x\,dx = \int\tan^2 3x\tan 3x\,dx = \int(\sec^2 3x - 1)\tan 3x\,dx$

$= \int\sec^2 3x\tan 3x - \tan 3x\,dx \leftarrow$ $\quad \int\tan 3x\,dx = \frac13\ln|\sec 3x| + c$

$\int\sec^2 3x\tan 3x\,dx$ $\quad u = \tan 3x$ $\quad \frac{du}{dx} = \sec^2 3x\cdot 3.$ $\quad \int\sec^2 3x\,u\,\frac{dx}{du}\,du = \int\sec^2 3x\,u\,\frac{1}{3\sec^2 3x}\,du$

$= \frac13\int u\,du = \frac16 u^2 + c = \frac16\tan^2 3x + c$.

$\int\tan^3 3x\,dx = \frac16\tan^2 3x + \frac13\ln|\sec 3x| + c$

Q17 $\int\frac{7-x}{(x-2)(x+1)}\,dx$ $\quad \frac{7-x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$.

$x = -1: \quad 8 = -3B \quad B = -3/8$
$x = 2: \quad 5 = 3A \quad A = 3/5$

$\int\frac{3/5}{x-2}\,dx + \int\frac{-3/8}{x+1}\,dx = \frac35\ln|x-2| - \frac38\ln|x+1| + c$