Sample final solutions

Q1

$$y = \cos^{-1}(x)$$

$$\cos y = x$$

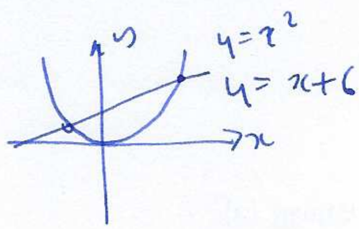
$$-\sin y \cdot \frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$$

Q2

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-1/x^3)^2}} \cdot \frac{3}{x^4} \qquad \text{at } x=1, \left.\frac{dy}{dx}\right|_1 = 3$$

tangent line $y = 3(x-1)$

Q3



intersections: $x^2 = x+6$

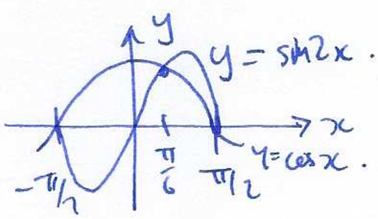
$$x^2 - x - 6 = 0$$

$$(x-3)(x+2) = 0 \qquad x = 3, -2.$$

$$\int_{-2}^{3} x+6-x^2 \, dx = \left[\tfrac{1}{2}x^2 + 6x - \tfrac{1}{3}x^3\right]_{-2}^{3}$$

$$= \frac{9}{2} + 18 - 9 - \left(2 - 12 + \frac{8}{3}\right) = 20\tfrac{5}{6}$$

Q4



find intersections: $\sin 2x = \cos x$

$$2\sin x \cos x = \cos x$$

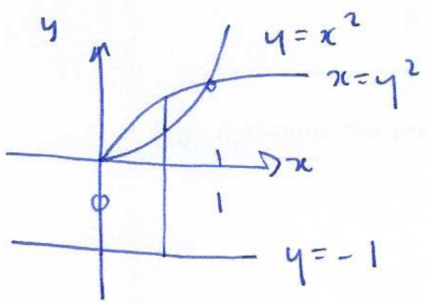
$$\cos x\,(2\sin x - 1) = 0$$

$\cos x = 0,\ x = \pm\frac{\pi}{2}$ $\qquad \sin x = \frac{1}{2} \quad x = \frac{\pi}{6}$

$$\int_{-\frac{\pi}{2}}^{\pi/6} \cos x - \sin 2x \, dx + \int_{\pi/6}^{\pi/2} \sin 2x - \cos x \, dx = \left[+\sin x + \tfrac{1}{2}\cos 2x\right]_{-\pi/2}^{\pi/6} + \left[-\tfrac{1}{2}\cos 2x - \sin x\right]_{\pi/6}^{\pi/2}$$

$$= \frac{1}{2} + \frac{1}{4} - \left(-1 - \frac{1}{2}\right) + \left(\frac{1}{2} - 1\right) - \left(-\frac{1}{4} - \frac{1}{2}\right) = \cancel{\tfrac{11}{4}}\ 2\tfrac{1}{2}$$

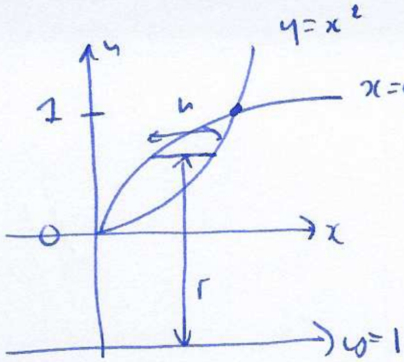
Q5



discs: $\displaystyle\int_0^1 \pi\left((\sqrt{x}+1)^2 - (x^2+1)^2\right) dx$

$$\pi \int_0^1 x + 2\sqrt{x} + \cancel{1} - x^4 - 2x^2 - \cancel{1} \, dx$$

$$\pi\left[\tfrac{1}{2}x^2 + \tfrac{4}{3}x^{3/2} - \tfrac{1}{5}x^5 - \tfrac{2}{3}x^3\right]_0^1 = \frac{29}{30}\pi$$

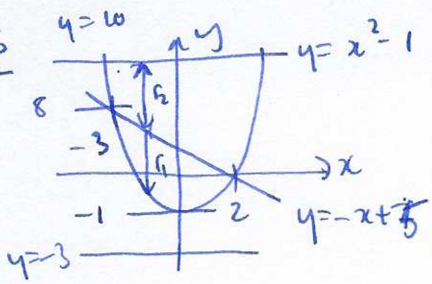


shells: $\int_0^1 2\pi r h\,dy = \int_0^1 2\pi (y+1)(\sqrt{y}-y^2)\,dy$

$= 2\pi \int_0^1 y^{3/2} - y^3 + y^{1/2} - y^2\,dy = 2\pi\left[\frac{2}{5}y^{5/2} - \frac{1}{4}y^4 + \frac{2}{3}y^{3/2} - \frac{y^3}{3}\right]_0^1$

$= \frac{29\pi}{30}$

Q6



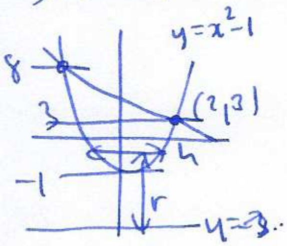
find intersection points: $x^2-1=-x+5$ $\quad x^2+x-6=0$

$(x-2)(x+3)=0 \quad x=2,-3 \quad (2,3)\ (-3,8)$

a) $y=10$, discs $\cancel{\int_{-3}^{2} \pi\left((x^2-1)^2 - (-x+5)^2\right) dx}$

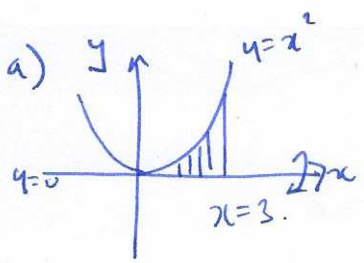
$\int_{-3}^{2} \pi(r_1^2 - r_2^2)\,dx = \int_{-3}^{2} \pi\left((x^2-1-10)^2 - (-x+5-10)^2\right) dx$

b) $y=-3$, shells. $\int_{-1}^{8} 2\pi r h\,dy = \int_{-1}^{3} 2\pi (2\sqrt{y+1})(y+3)\,dy$



$+ \int_{3}^{8} 2\pi (y+3)(-y+5-\sqrt{y+1})\,dy$

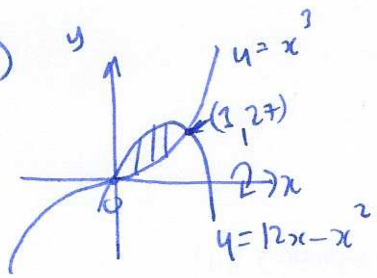
Q7 a)



discs: $\int_0^3 \pi r^2\,dx = \int_0^3 \pi (x^2)^2\,dx = \int_0^3 \pi x^4\,dx$

$= \left[\pi \frac{1}{5} x^5\right]_0^3 = \frac{243\pi}{5}$

b)



find intersection points: $x^3 = 12x - x^2 \quad x^3 + x^2 - 12x = 0$

$x(x^2+x-12)=0 \quad x(x+4)(x-3)=0 \quad x=-4,0,3.$

discs: $\int_0^3 \pi(r_1^2 - r_2^2)\,dx = \int_0^3 \pi\left((12x-x^2)^2 - (x^3)^2\right) dx$

$= \pi\int_0^3 144x^2 - 24x^3 + x^4 - x^6\,dx = \pi\left[\cancel{\tfrac{144}{3}}^{\,48} x^3 - 6x^4 + \frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^3 = \frac{19116}{5}\pi$

Q8 a) $y = \ln(\cos x)$ arc length : $\int_0^{\pi/4} \sqrt{1+y'^2}\, dx$

$\frac{dy}{dx} = \frac{1}{\cos x} \cdot -\sin x$

$= -\tan x$

$= \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx$

$= \left[\ln|\sec x + \tan x| \right]_0^{\pi/4} = \ln\left| \frac{\sqrt{2}}{2} + 1 \right|$

b) $x = \frac{1}{2}(e^y + e^{-y})$

$\frac{dx}{dy} = \frac{1}{2}(e^y - e^{-y})$

arc length : $\int_0^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 \sqrt{1+\left(\frac{1}{4}(e^y - e^{-y})^2\right)}\, dy$

$= \int_0^2 \sqrt{1 + \frac{1}{4}e^{2y} - \frac{1}{2} + \frac{1}{4}e^{-2y}}\, dy = \int_0^2 \sqrt{\frac{1}{4}e^{2y} + \frac{1}{2} + \frac{1}{4}e^{-2y}}\, dy = \int_0^2 \sqrt{\left(\frac{1}{2}(e^y + e^{-y})\right)^2}\, dy$

$= \int_0^2 \frac{1}{2}(e^y + e^{-y})\, dy = \left[\frac{1}{2}(e^y - e^{-y}) \right]_0^2 = \frac{1}{2}(e^2 - e^{-2}) - 0$.

Q9 $y = x^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$

arc length : $\int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{1 + \frac{9}{4}x}\, dx$

$= \left[\left(1 + \frac{9}{4}x\right)^{3/2} \cdot \frac{2}{3} \cdot \frac{4}{9} \right]_0^3 = \frac{8}{27}\left(\left(\frac{31}{4}\right)^{3/2} - 1 \right)$.