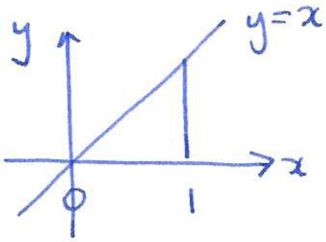
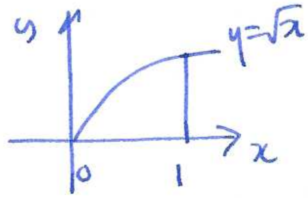
so $\int_a^t f(x)\,dx = F(t) - F(a)$ $\square$.

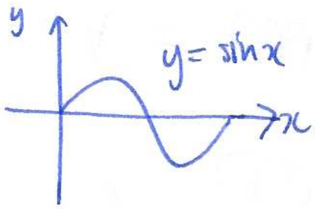
Examples




$$\int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$

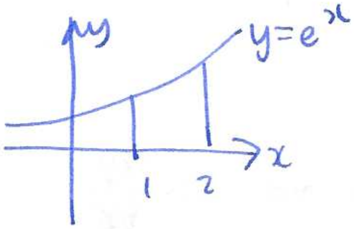



$$\int_0^1 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^1 = \tfrac{2}{3} - 0 = \tfrac{2}{3}$$

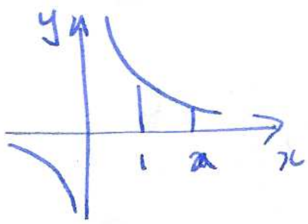



$$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = -\cos(\pi) + \cos(0) = -(-1) + 1 = 2$$

$\int_0^{2\pi} \sin x\,dx = 0$ ! why?




$$\int_1^2 e^x\,dx = \left[e^x\right]_1^2 = e^2 - e$$




$$\int_1^a \frac{1}{x}\,dx = \left[\ln|x|\right]_1^a = \ln|a| - \ln|1| = \ln|a|.$$

Observations

① choice of antiderivative doesn't matter, let $F(x)$, and $F(x)+c$ be antiderivatives for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

$$= F(b) + c - (F(a) + c) = F(b) - F(a)$$

② $\int_a^t f(x)\,dx$ is a function of $t$! (and $a$) but not $x$ (called dummy variable)

i.e. $\int_a^t f(x)\,dx = \int_a^t f(y)\,dy$. If you want a function of $x$ write $\int_a^x f(t)\,dt$.

## §5.4 Fundamental theorem of calculus II

Thm (FTC ②) Let $f(x)$ be a cts function on $[a,b]$, then $A(x) = \int_a^x f(t)\,dt$ is an antiderivative for $f(x)$, ie. $A'(x) = f(x) = \frac{dA}{dx}$, so $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Furthermore, $A(a) = 0$ $\square$.

## §5.6 Substitution/change of variable

"reverse chain rule for integration"

recall : chain rule : $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

substitution | change of variables

$\int f(x)\,dx$ , $x(u)$

$$\boxed{\int_{x=a}^{x=b} f(x)\,dx = \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\,\frac{dx}{du}\,du}$$

remember, three things to change : function, differential, limits.

mnemonic : cancelling fractions ~~$dx = \frac{du}{dx}\,dx$~~ $dx = \frac{dx}{du}\,du$

why does this work?

$\int f(x)\,dx$, $x(u)$ set $F(x) = \int f(x)\,dx$, so $F'(x) = f(x)$

$$\int f(x)\,dx = F(x) \overset{\text{diff}}{\underset{\text{wrt } u}{\rightsquigarrow}} \frac{d}{du}(F(x(u))) = F'(x(u)) \cdot x'(u)$$

$$\overset{\text{integrate}}{\underset{\text{wrt } u}{\rightsquigarrow}} = \int f(x(u))\,x'(u)\,du = \int f(x(u))\,\frac{dx}{du}\,du \quad \square.$$

useful fact : $\frac{dx}{du} = 1 \Big/ \frac{du}{dx}$

Examples $\int_{x=0}^{x=1} e^{-7x}\,dx$. set $u = -7x$

$\frac{du}{dx} = -7$

$$\int_{u=0}^{u=-7} e^u \frac{dx}{du}\,du = \int_0^{-7} e^u \cdot \frac{1}{7}\,du = -\frac{1}{7}\left[e^u\right]_0^{-7} = -\frac{1}{7}(e^{-7} - 1)$$