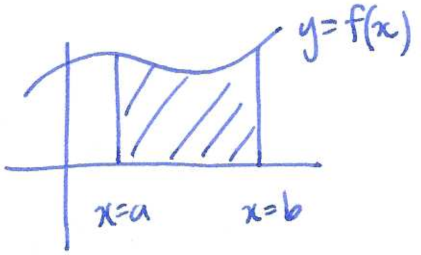
§5.2 Definite integral

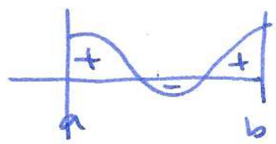


$\int_a^b f(x)\,dx =$ area under the curve $y=f(x)$ between $x=a$ and $x=b$

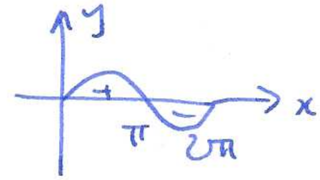
Formal definition: Riemann sum $R(f,P,c)$

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Note: signed area!

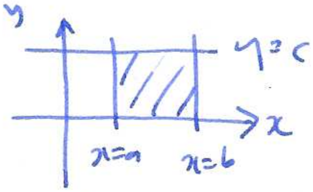


so $\int_0^{2\pi} \sin(x)\,dx = 0$



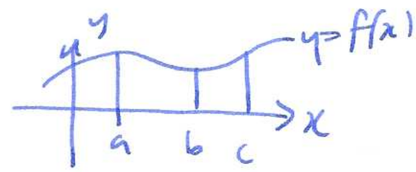
Useful properties

$$\int_a^b c\,dx = c(b-a)$$



$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

adjacent intervals: $\int_a^b f(x)dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$



0-length interval: $\int_a^a f(x)\,dx = 0$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)dx$

comparisons: $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

§5.3 Anti-derivatives

Defn A function $F(x)$ is an anti-derivative for $f(x)$ if $F'(x) = f(x)$

General antiderivative

Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x)+c$ for some $c \in \mathbb{R}$

$\int cf(x)\,dx = c\int f(x)\,dx$ □

Warning: no product / quotient / chain rule!

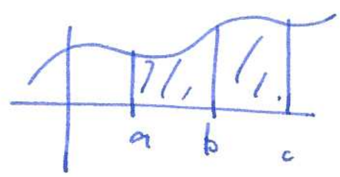
Useful integrals $\int \sin(x)\,dx = -\cos x + c$ $\qquad \int e^x\,dx = e^x + c$

$\int \cos(x)\,dx = \sin x + c$

with limits / definite integrals: $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$



0-length intervals: $\int_a^a f(x)\,dx = 0$

adjacent intervals: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

comparisons: if $f(x) \le g(x)$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

§5.3 Anti-derivatives

Defn A function $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$.

Example if $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an anti-derivative

check: $F'(x) = \frac{1}{3}\cdot 3x^2 = x^2$ ✓

Note: $F(x) = \frac{1}{3}x^3 + 4$ is also an antiderivative. $\left(\frac{1}{3}x^3+4\right)' = x^2$

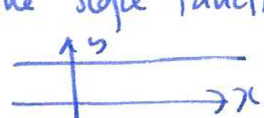
General antiderivative

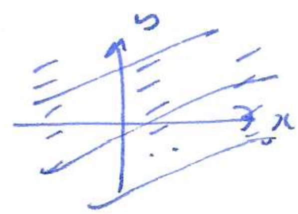
Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x)+c$ for some $c \in \mathbb{R}$.

Proof suppose $F(x)$ and $G(x)$ are antiderivatives for $f(x)$, then $F'(x) = f(x)$ and $G'(x) = f(x)$, so $(F(x)-G(x))' = f(x)-f(x) = 0$ so $F(x)-G(x) = c$ constant function □.

Picture: $f(x)$ gives the slope function for $F(x)$

Examples $f(x) = c$





$F'(x) = c$

$F(x) = cx + d$

<u>Examples</u> $\int x^2 + \frac{1}{x} + \sin x \, dx = \frac{1}{3}x^3 + \ln|x| - \cos(x) + c$

<u>Observation</u>: every rule for differentiation gives a rule for integration.

<u>Alternate view</u>: we can think of finding the indefinite integral as finding a function given its slopes, i.e. its derivative. This is an example of solving a differential equation $\frac{dy}{dx} = f(x)$. In general there is a family of solutions $F(x) + c$, but if we know the value of the solution we want at $x=0$ (sometimes called an initial condition) this gives a particular solution.

<u>Example</u>: motion under gravity, acceleration $a(t) = x''(t) = -g$ (constant)

velocity $v(t) = x'(t) = -gt + c$

if $v(0) = v_0$, then $v(t) = -gt + v_0$

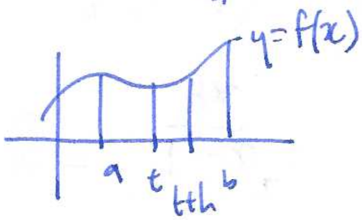
position $x(t) = -\frac{1}{2}gt^2 + v_0 t + c$

if $x(0) = x_0$, then $x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0$

§5.4 Fundamental theorem of calculus

<u>Thm</u> (FTC ①) suppose $f(x)$ is cts on $[a,b]$ and $F(x)$ is an antiderivative for $f(x)$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$

<u>intuition</u>:



consider $\int_a^t f(x)\,dx$

Q: what is the rate of change wrt t?

$$\frac{d}{dt}\int_a^t f(x)\,dx = \lim_{h\to 0} \frac{\int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx}{h} = \lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(x)\,dx$$

$$\approx \frac{\text{area of rectangle } f(t)\times h}{h} = f(t)$$

i.e. $\int_a^t f(x)\,dx$ is an antiderivative for $f(x)$, so $\int_a^t f(x)\,dx = F(t) + c$

Q: what is the constant? $t = a$: $\int_a^a f(x)\,dx = 0 = F(a) + c \Rightarrow c = -F(a)$