approximate with 4 rectangles: area ≈ sum of $\underbrace{\text{area of rectangles}}_{\text{width} \times \text{height}}$

$$= \frac{1}{4}f(0) + \frac{1}{4}f\left(\frac{1}{4}\right) + \frac{1}{4}f\left(\frac{2}{4}\right) + \frac{1}{4}f\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{2}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{4}\sum_{i=0}^{3} f\left(\frac{i}{4}\right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) = \frac{1}{4}\left(\frac{6}{4}\right) = \frac{3}{8} \approx 0.375$$

approximate with n rectangles.

$$f(0)\frac{1}{n} + f\left(\frac{1}{n}\right)\frac{1}{n} + f\left(\frac{2}{n}\right)\frac{1}{n} + \cdots + f\left(\frac{n-1}{n}\right)\frac{1}{n} = \sum_{i=0}^{n-1}\frac{1}{n}f\left(\frac{i}{n}\right)$$

$$= \frac{1}{n}\left(f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right) \qquad = \sum_{i=0}^{n-1}\frac{1}{n}\cdot\frac{i}{n}$$

$$= \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right) \qquad = \frac{1}{n^2}\sum_{i=0}^{n-1} i$$

$$= \frac{1}{n^2}\left(1 + 2 + 3 + \cdots + (n-1)\right)$$

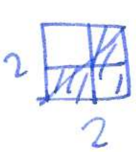
<u>claim</u>: $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

<u>Proof</u> ① induction: assume true for k: $S_k = 1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$

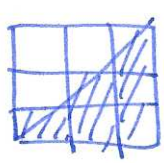
$$S_{k+1} = \underbrace{1 + 2 + \cdots + k}_{S_k} + k+1 = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left(\frac{1}{2}k + 1\right) = \frac{1}{2}(k+1)(k+2) \checkmark$$

finally check $k=1$ $\quad S_1 = \frac{1}{2}\cdot 1(2) = 1$ $\quad \square$

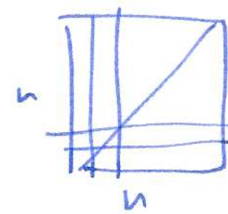
②



$\frac{1}{2}(2)^2 + \frac{1}{2}\cdot 2$



$\frac{1}{2}(3)^2 + \frac{1}{2}(3)$



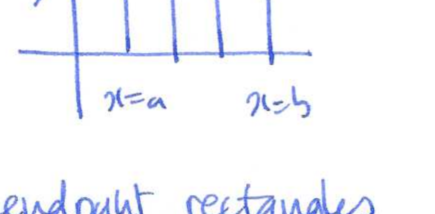
$\frac{1}{2}n^2 + \frac{1}{2}n = \frac{1}{2}n(n+1)$

so approximate area is

$$\frac{1}{n^2}\left(1 + 2 + \cdots + (n-1)\right) = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{1}{2}\frac{n^2-n}{n^2} = \frac{1}{2}\left(1 - \frac{1}{n}\right) \to \frac{1}{2}$$

as $n \to \infty$.

<u>Notation</u>

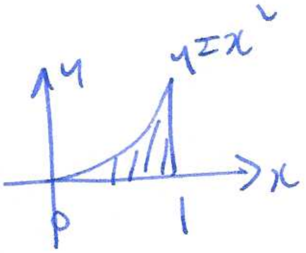


N rectangles of equal width then $\Delta x = \dfrac{b-a}{N}$

left endpoint rectangles $\quad L_N = \sum_{i=0}^{N-1} f(a + i\Delta x)\Delta x$

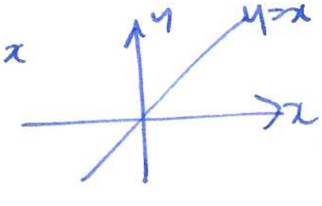
right endpoint rectangles $\qquad R_N = \sum_{i=1}^{N} f(a+i\Delta x)\Delta x$

midpoint rectangles $\qquad M_N = \sum_{i=1}^{N} f(a+(i-\tfrac{1}{2})\Delta x)\Delta x$



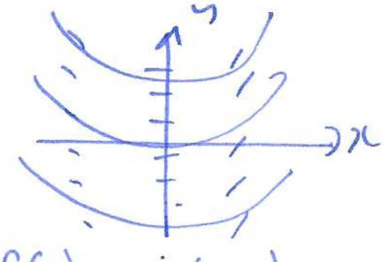
$1+2^2+3^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ .... need better way...

$f(x) = x$



$F(x) = \frac{1}{2}x^2+c$

$F'(x) = f(x)$



<u>Examples</u> : find the general antiderivative to $f(x) = \sin(4x)$

guess: $\frac{d}{dx}(\cos(4x)) = -\sin(4x)\cdot 4$

so $\frac{d}{dx}\left(-\frac{1}{4}\cos(4x)\right) = -\frac{1}{4}\cdot -\sin(4x)\cdot 4 = \sin(4x)$

so $F(x) = -\frac{1}{4}\cos(4x)+c$

<u>Notation</u> : indefinite integral

$\int f(x)\,dx = F(x)+c$ $\quad$ means: $F(x)+c$ is the general antiderivative for $f(x)$

<u>Thm</u> $\int x^n\,dx = \frac{1}{n+1}x^{n+1}+c$ $\quad$ for $n \neq -1$

<u>Proof</u> $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) = \frac{1}{n+1}(n+1)x^n = x^n$ $\square$.

<u>Thm</u> $\int \frac{1}{x}\,dx = \ln|x|+c$

<u>Proof</u> $(x>0)$ $\frac{d}{dx}(\ln(x)+c) = \frac{1}{x}$.

<u>Thm</u> (sums and constant multiples)

$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$