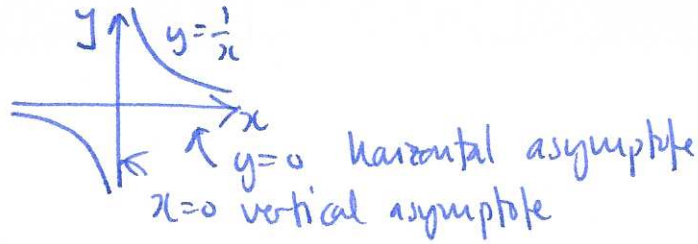
Asymptotes

Example $y = \frac{1}{x}$



Defn $x = c$ is a <u>vertical asymptote</u> if $\lim\limits_{x \to c^{\pm}} f(x) = \pm\infty$

$y = c$ is a <u>horizontal asymptote</u> if $\lim\limits_{x \to \pm\infty} f(x) = c$

Observation: rational functions $\frac{p(x)}{q(x)}$ have horizontal asymptotes if $\deg p \leq \deg q$.

Example $f(x) = \dfrac{x^2+x+1}{3x^2+2} \sim \dfrac{x^2}{3x^2} \to \dfrac{1}{3}$ as $x \to \infty$.

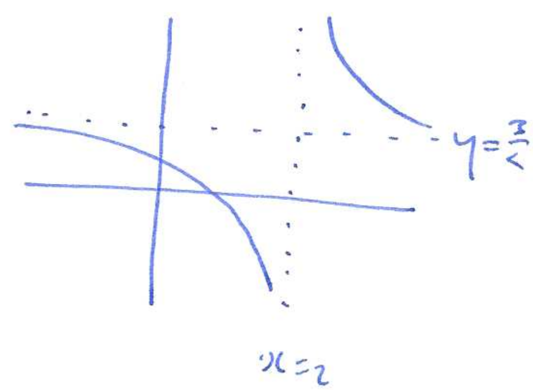
Example sketch graph of $f(x) = \dfrac{3x+2}{2x-4}$

① find vertical asymptotes $\Leftrightarrow$ denominator zero: $2x-4=0$ $\quad x=2$

② find $f'(x) = \dfrac{(2x-4)3-(3x+2).2}{(2x-4)^2} = \dfrac{-16}{(2x-4)^2} = \dfrac{-4}{(x-2)^2}$ ← always $-$ve. (f decreasing).

no critical points except vertical asymptote at $x=2$

③ $f''(x) = \dfrac{8}{(x-2)^3}$ $\quad$ $+$ve $\quad x>2$ concave up
$\quad$ $-$ve $\quad x<2$ concave down.



④ horizontal asymptotes: $\dfrac{3x+2}{2x-4} \sim \dfrac{3x}{2x} = \dfrac{3}{2}$.

behaviour near asymptote

	$x < -\frac{2}{3}$	$-\frac{2}{3} < x < 2$	$x > 2$
$3x+2$	$-$	$+$	$+$
$2x-4$	$-$	$-$	$+$
$f(x)$	$+$	$-$	$+$

Example sketch graph of $f(x) = \dfrac{x}{\sqrt{x^2+1}} = x\,(x^2+1)^{-1/2}$

① vertical asymptotes : none

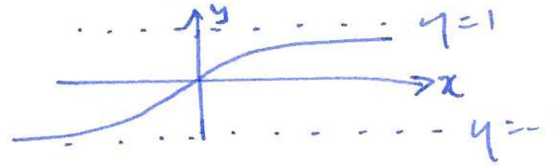
② find $f'(x) = \dfrac{\sqrt{x^2+1} - x \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x}{x^2+1} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = (x^2+1)^{3/2} > 0$

$\Rightarrow$ f increasing, no critical points.

③ $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x$ point of inflection at $x=0$ +ve for $x<0$, −ve for $x>0$

④ horizontal asymptotes $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} \overset{l'H}{=} \lim\limits_{x\to\infty} \dfrac{1}{\frac{1}{2}(x^2+1)^{-1/2} \cdot 2x} = \lim\limits_{x\to\infty} \sqrt{1+1/x^2} = 1$

$\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{-x}{\sqrt{x^2+1}} = -1.$



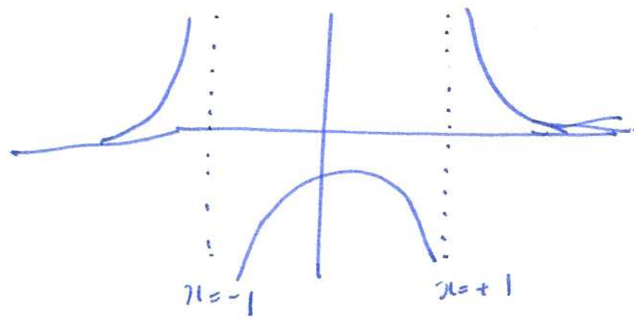
<u>Example</u> $f(x) = \dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)}$

① vertical asymptotes at $x = \pm 1$

② $f'(x) = -(x^2-1)^{-2} \cdot 2x$ critical point at $x=0$

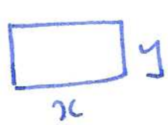
③ $f''(x) = \dfrac{6x^2+2}{(x^2-1)^3}$

④ etc.



<u>§4.7 Optimization</u>

<u>Example</u> A piece of wire of length L is bent into a rectangle. What's the largest possible area?



area $A = xy$
length $L = 2x+2y$ $\Big\}$ $y = \frac{L}{2} - x$

$A = x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2$

$A'(x) = \frac{L}{2} - 2x$ critical point $x = \frac{L}{4}$ (local max)

so max area of $\frac{L^2}{16}$ occurs when $x = y = \frac{L}{4}$ (square).

<u>Example</u> what shape of cylindrical can minimizes surface area, if you want total volume to be $1\,ft^3$?