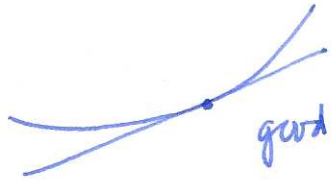
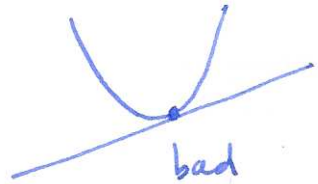
Observation: when is the linear approximation a good approximation?



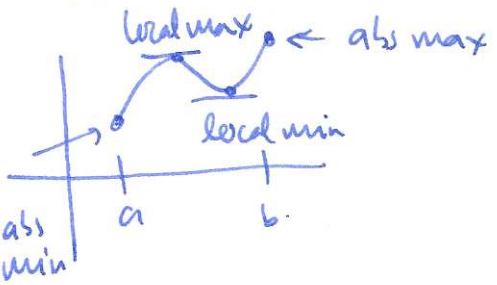
$f''(x)$ small



$f''(x)$ large

§4.2 Extreme values

suppose $f(x)$ is defined on a closed interval $[a,b]$



Def^n^ $f(c)$ is the absolute max if $f(c) \geq f(x)$ for all $x \in [a,b]$

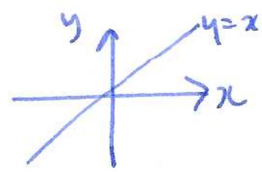
$f(c)$ is the absolute min if $f(c) \leq f(x)$ for all $x \in [a,b]$

note: Q: where is the local / abs max/min ← want x-value

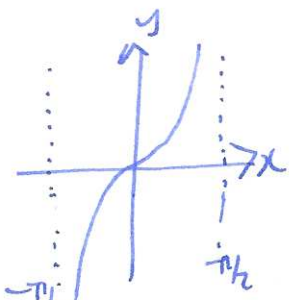
Q: what is the local / abs max/min ← want y-value.

warning some functions do not have any max/min

Examples: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x$



$f: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, $x \to \tan(x)$



Thm If $f(x)$ is continuous on a closed bounded interval then $f(x)$ has both an absolute max and an absolute min

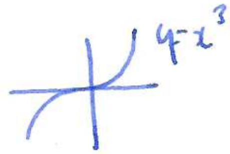
Def^n^ $f(x)$ has a local max at $x = c$ if there is a small interval containing c s.t. $f(c)$ is an abs maximum on this interval

$f(x)$ has a local min at $x = c$ if there is a small interval containing c s.t. $f(c)$ is an abs minimum on this interval.

Def^n^ we say that c is a critical point if $f'(c) = 0$ (or undefined)

<u>warning</u> $f'(c)=0 \not\Rightarrow$ local max or min

<u>Example</u> $y=x^3$



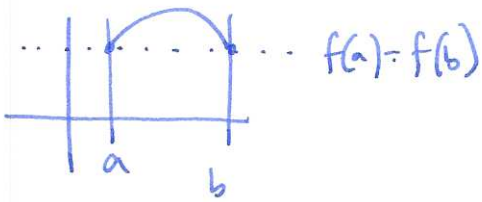
$f'(x)=3x^2$ $f'(0)=0$ but $x=0$ not local max or min

• How to find the absolute max or min of a differentiable function on a closed interval $[a,b]$: ① find critical points, evaluate function there
② check endpoints.

<u>Example</u> ① find abs max/min of $2x^3-15x^2+24x+7$ on $[0,3]$
② x^2-8 on $[1,4]$ ③ $\cos(x)\sin(x)$ on $[0,\pi]$.

<u>Thm</u> (Rolle's Thm) Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$ then there is a $c\in(a,b)$ s.t. $f'(c)=0$

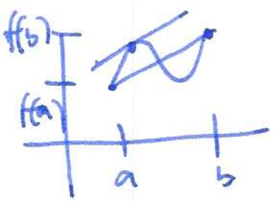
<u>Proof</u>



• if there is a local max/min at c then $f'(c)=0$
• if no local max/min, then $f(c)=$ const $=f(a)=f(b)$ so $f'(x)=0$ for all $c\in(a,b)$ □

§4.3 first derivative test

<u>Thm</u> (Mean value theorem) (MVT) Suppose f is continuous on $[a,b]$ and differentiable on (a,b), then there is a $c\in(a,b)$ s.t. $f'(c)=\frac{f(b)-f(a)}{b-a}$, i.e. there is a point where the slope is equal to the average rate of change.



<u>Proof</u> (Rolle's Thm turned sideways)

<u>Corollary</u> If $f(x)$ is differentiable, and $f'(x)=0$, then $f(x)=c$ (constant)

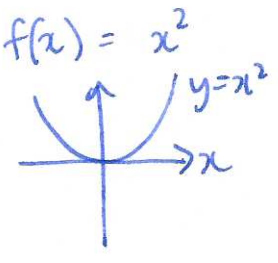
<u>Proof</u> suppose there is a,b with $f(a)\neq f(b)$, then $\exists\, c\in(a,b)$ with $f'(c)=\frac{f(b)-f(a)}{b-a}\neq 0$ □.

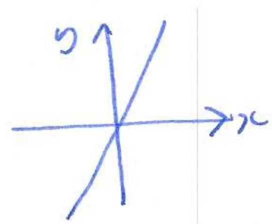
Monotonicity suppose f is differentiable on (a,b)

If $f'(x) > 0$ for all $x \in (a,b)$ then f is increasing on (a,b)

$f'(x) < 0$ decreasing

Example ① $f(x) = x^2$ $f'(x) = 2x$ increasing on $(0,\infty)$ decreasing on $(-\infty,0)$





② $f(x) = x^2 - 2x - 3$

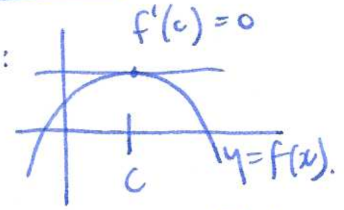
$f'(x) = 2x - 2$

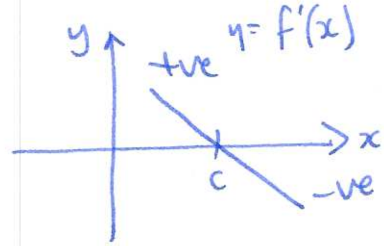
$f'(x) > 0$ where $2x - 2 > 0$

$x > 1$

First derivative test

local max :

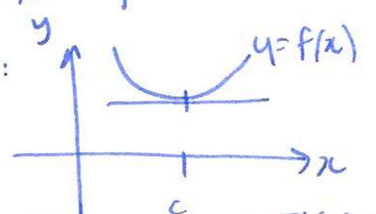


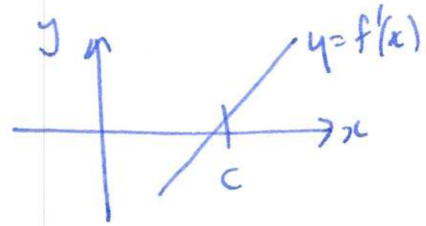


$f'(x) > 0$ $f'(x) < 0$

$f'(x)$ goes from positive to negative $\Rightarrow$ local max.

local min :





$f'(x) < 0$ $f'(x) > 0$

$f'(x)$ goes from negative to positive $\Rightarrow$ local min

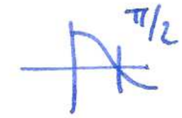
Thm **First derivative test** If $f(x)$ is differentiable and $f'(c) = 0$ then if $f'(x)$ changes from +ve to −ve at c $\Rightarrow$ c local max

−ve to +ve $\Rightarrow$ c local min

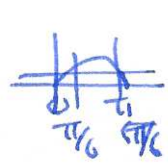
Example classify critical points of $f(x) = \cos^2 x + \sin x$ on $[0,\pi]$.

find critical points: $f'(x) = -2\cos(x)\sin x + \cos x$

solve: $f'(x) = 0$ $\cos(x)(1 - 2\sin x) = 0$ $\cos x = 0$



$1 - 2\sin x = 0 \Leftrightarrow \sin x = \frac{1}{2}$



$x = \frac{\pi}{6}, \frac{5\pi}{6}$

so critical points are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$