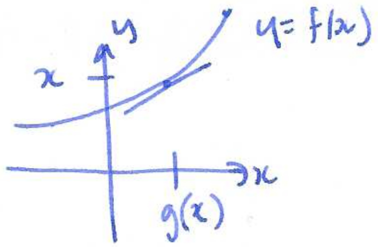
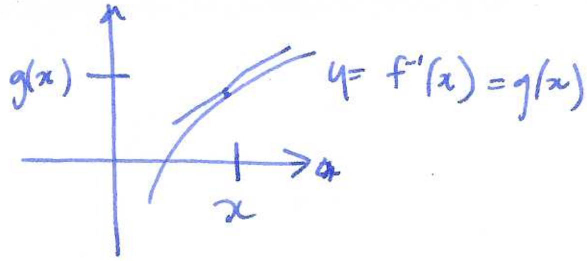
recall :



$\longrightarrow$

reflect in $y = x$



$\longleftarrow$ slope at x: $g'(x)$

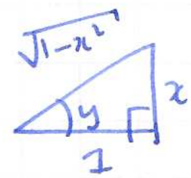
$\frac{+1}{\text{slope}}$ at $g(x)$, i.e $\dfrac{1}{f'(g(x))}$

Application : derivatives of inverse trig functions.

Thm $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ $\quad \dfrac{d}{dx}\left(\cos^{-1}(x)\right) = \dfrac{-1}{\sqrt{1-x^2}}$

Proof (of $\sin^{-1}(x)$) :

$$y = \sin^{-1}(x)$$



$$\sin(y) = x$$

$$\cos(y)\, y' = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}} \quad \square.$$

Thm $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$ $\quad \dfrac{d}{dx}\left(\cot^{-1}(x)\right) = \dfrac{-1}{1+x^2}$

$\dfrac{d}{dx}\left(\sec^{-1}(x)\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$ $\quad \dfrac{d}{dx}\left(\csc^{-1}(x)\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

Proof (of $\tan^{-1}(x)$) :

$$y = \tan^{-1}(x)$$

$$\tan(y) = x$$

$$\sec^2(y)\, y' = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \quad \square$$

Example $\dfrac{d}{dx}\left(\tan^{-1}(e^{2x})\right) = \dfrac{1}{1+(e^{2x})^2} \cdot e^{2x} \cdot 2.$

§3.9 Derivatives of exponentials and logs

recall : $f(x) = b^x$ then $f'(x) = m_b b^x$
$f(x) = e^x$ then $f'(x) = e^x$

Thm: $f(x) = b^x$ then $f'(x) = \ln(b)\, b^x$

Proof $f(x) = b^x = e^{\ln(b)x}$ $\quad f'(x) = e^{x\ln(b)} \cdot \ln(b) = b^x \cdot \ln(b)$ $\square$.

Example $f(x) = 3^{4x} = (3^4)^x$ $\quad f'(x) = \ln(3^4)\,(3^4)^x = 4\ln(3) \cdot 3^{4x}$.

Thm: $f(x) = \ln(x)$ then $f'(x) = \frac{1}{x}$.

Proof $y = \ln(x)$ $\quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$. $\square$.

$e^y = x$

$e^y \cdot y' = 1$

Example ① $f(x) = \log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ $\quad f'(x) = \dfrac{1}{x\ln(b)}$.

② $f(x) = x\ln(x)$

$f'(x) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$

Hyperbolic trig functions $\quad \sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\quad \cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\frac{d}{dx}(\sinh(x)) = \cosh x$ $\quad \frac{d}{dx}(\cosh(x)) = \sinh(x)$ $\quad \boxed{\cosh^2 x - \sinh^2 x = 1}$

check : $(\sinh(x))' = \frac{1}{2}(e^x - e^{-x})' = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$

$\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ so $\frac{d}{dx}(\tanh(x)) = \dfrac{1}{\cosh^2 x}$, etc

inverse functions : $\frac{d}{dx}(\sinh^{-1}(x)) = \dfrac{1}{\sqrt{x^2+1}}$

$\frac{d}{dx}(\cosh^{-1}(x)) = \dfrac{1}{\sqrt{x^2-1}}$ $\quad \frac{d}{dx}(\tanh^{-1}(x)) = \dfrac{1}{1-x^2}$.

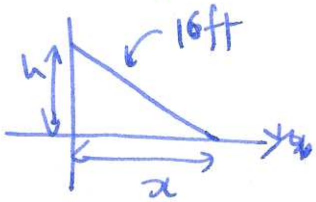
§3.10 Related rates

Example ① ballon :  $V = \frac{4}{3}\pi r^3$ inflate ballon: $V(t) = \frac{4}{3}\pi r(t)^3$.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ so if $\frac{dV}{dt} = 1\ ft^3/min$ and $r = 1\ ft$: $1 = 4\pi \cdot 1 \cdot \frac{dr}{dt}$

then $\frac{dr}{dt} = \frac{1}{4\pi}$ ft/min.

② falling ladder



$x(t)$ distance of foot of ladder from wall
$h(t)$ height of top of ladder against wall

Q: if $\frac{dx}{dt} = 3\ ft/s$ what is $\frac{dh}{dt}$?

true for all t : $x^2 + h^2 = 16^2$ suppose $h = 5$ at $t = 0$

$\frac{dx}{dt} = 3$. true for same t : $h(0) = 5$.

consider : $x^2 + h^2 = 16^2 \iff (x(t))^2 + (h(t))^2 = 16^2$

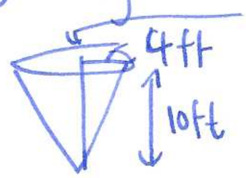
diff wrt t : $2x\,x' + 2h\,h' = 0$

$\frac{dx}{dt} = 3$ for all $t \Rightarrow x(t) = 3t + 5$ so $h(t) = \sqrt{16^2 - (3t+5)^2}$.

so $\frac{dh}{dt} = -\frac{x(t)}{h(t)} \cdot 3 = \frac{-3(3t+5)}{\sqrt{16^2 - (3t+5)^2}}$

hit ground at $5 + 3t = 16$, $t = 11/3$ $\frac{dh}{dt}\left(\frac{11}{3}\right) = \frac{-16}{\sqrt{16^2 - 16^2}}$ i.e. $\frac{dh}{dt} \to \infty$ as $t \to 11/3$.

③ filling a conical tank



$10 ft^3/min$

Q: how fast is the water rising when $h = 5ft$?

water in : $\frac{dV}{dt} = 10\ ft^3/min$

volume of cone: $V = \frac{1}{3}\pi h r^3$

need r in terms of h : $\frac{r}{h} = \frac{4}{10}$ i.e. $r = \frac{2}{5}h$

so $V = \frac{1}{3}\pi h \left(\frac{2}{5}h\right)^2 = \frac{4}{75}\pi h^3$.

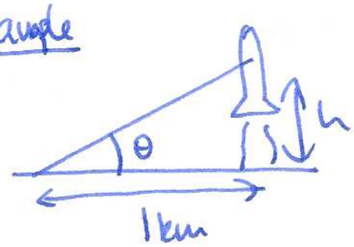
$\frac{dV}{dt} = \frac{12}{75}\pi h^2 \frac{dh}{dt}$ so when $h = 5$: $10 = \frac{12}{75}\pi (5)^2 \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{10}{4\pi}$ ft/min.

<u>advice</u> ① give things names

② write down relations between things and use implicit differentiation

③ plug in numbers as necessary.

<u>Example</u>

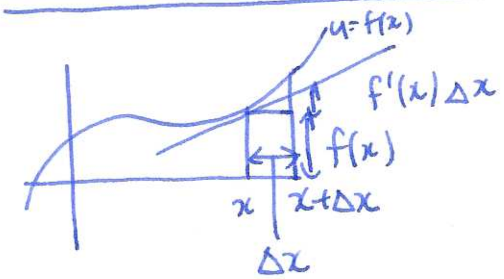


if angle is $\theta = \frac{\pi}{3}$ and rate of change is $\frac{d\theta}{dt} = \frac{1}{2}$ rad/min, how fast is the rocket going?

$$\frac{h}{1} = \tan\theta \qquad \frac{dh}{dt} = \sec^2\theta \frac{d\theta}{dt} \qquad \frac{dh}{dt} = \sec^2\left(\frac{\pi}{3}\right)\frac{1}{2}$$

$\simeq 0.1$ km/min

§4.1 Linear approximation



If $f(x)$ is differentiable at x, and Δx is small, then $f(x+\Delta x) \approx f(x) + f'(x)\Delta x$

so change in f is

$$\Delta f \approx f(x+\Delta x) - f(x)$$

$$\Delta f \approx f'(x)\Delta x$$

<u>Example</u> estimate $\sqrt{103}$

$f(x) = \sqrt{x} = x^{1/2}$ $\qquad f(100) = 10$ $\qquad$ so $\Delta f \approx f'(x)\Delta x = \frac{1}{20}\cdot 3$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(100) = \frac{1}{20}$

so $\sqrt{103} \approx 10 + \frac{3}{20} = 10.15$

<u>Example</u> pizza size: you make an 18" pizza. If the diameter is accurate to ± 0.4in, how much pizza do you gain/lose?

$A = \pi r^2 \qquad 2r = D \qquad A = \pi\left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}$

$$A'(D) = \frac{2\pi D}{4} = \frac{\pi D}{2}$$

$$\Delta A \approx A'(18)\Delta D = \frac{1}{2}\pi \cdot 18 \cdot 0.4 \approx 11\,\text{in}^2$$

<u>Q</u>: is this good or bad? absolute error = 11

percentage error = $\left|\frac{\text{absolute error}}{\text{actual value}}\right| \times 100 = \left(\frac{11}{\pi \cdot 18^2/4}\right) \times 100 \sim 4\%$.