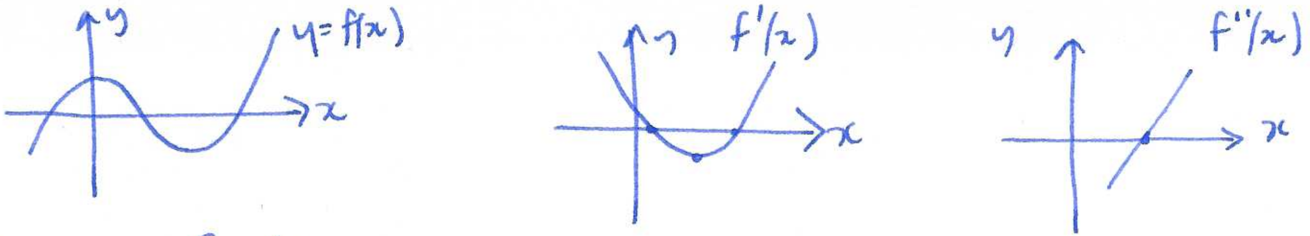
Example



§3.6 Trigonometric functions

Thm $\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$

Proof (for $\sin x$)

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

recall: $\sin(A+B) = \sin A \cos B - \cos A \sin B$

$$= \lim_{h\to 0} \frac{\sin x \cosh - \cos x \sinh - \sin x}{h} = \lim_{h\to 0} \sin x \cdot \frac{\cos(h)-1}{h} + \cos x \cdot \frac{\sinh}{h}$$

$$= \sin x \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{0} + \cos x \underbrace{\lim_{h\to 0} \frac{\sinh}{h}}_{1} = \cos x. \ \square.$$

Q: can this be right?

Example $f(x) = x\sin x$

$f'(x) = x\cos x + \sin x$

Thm $\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$

Proof (of $\tan x$) $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Example $\frac{d}{dx}(e^x \cos x) = e^x \cdot -\sin x + e^x \cos x.$

§3.7 Chain rule

Composition of functions: $f(g(x)) = (f \circ g)(x)$

Examples e^{4x}, $\sin^2(x)$, etc.

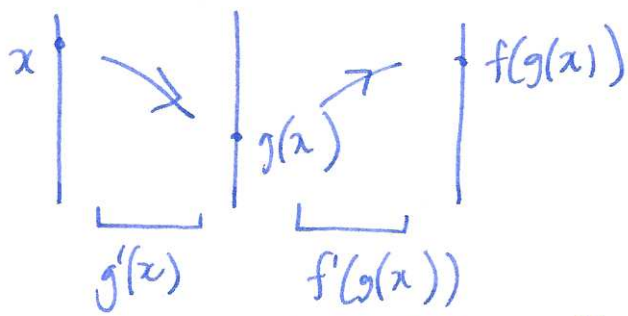
Thm If f, g differentiable functions, then $f \circ g$ is differentiable and

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

mnemonic: $\quad [f(g(x))]' = \text{outside}'(\text{inside}) \cdot \text{inside}'$

note: $f(g(x))$

$$\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$$
$$x \longmapsto g(x) \longmapsto f(g(x))$$



Examples ① $e^{4x} = f(g(x))$ where $f(x) = e^x \qquad g(x) = 4x$

$f'(x) = e^x \qquad g'(x) = 4$

so $(e^{4x})' = f'(g(x)).f(x) = e^{4x}.4$

② $\sin^2(x) = f(g(x))$ where $f(x) = x^2 \qquad g(x) = \sin(x)$

$f'(x) = 2x \qquad g'(x) = \cos(x)$

so $(\sin^2(x))' = 2(\sin(x)).\cos(x)$

③ $\sqrt{x^3+1}$ etc.

Alternate notation $\quad f(g(x)) \longleftrightarrow f(u), \quad u = g(x)$

$$\frac{df}{dx} = f'(u)\frac{du}{dx} = \frac{df}{du}\frac{du}{dx} \qquad \boxed{\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}}$$

mnemonic: "cancelling fractions"

Examples $\cos(x^2)$, $e^{\sqrt{x}}$, $\sin\left(\frac{\pi x}{180}\right)$, $\sqrt{x+\sqrt{x^2+1}}$.

Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \qquad [\text{answer should be } f'(g(x)).g'(x)]$$

write this as: $\lim_{h\to 0} \dfrac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \cdot \dfrac{g(x+h)-g(x)}{h}$

set $k = g(x+h) - g(x)$, as g is cts $h \to 0 \Rightarrow k \to 0$

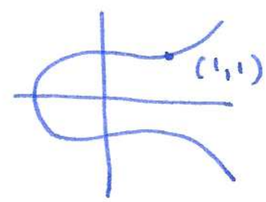
so $= \lim_{k\to 0} \dfrac{f(g(x)+k) - f(g(x))}{k} \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = f'(g(x)) \cdot g'(x)$ $\square$

<u>More examples</u> $\dfrac{d}{dx}\left(g(x)^n \right) = n (g(x))^{n-1} \cdot g'(x)$

$\dfrac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} \cdot g'(x)$

$\dfrac{d}{dx}\left(f(ax+b) \right) = a f'(ax+b)$

<u>§3.8 Implicit differentiation</u>



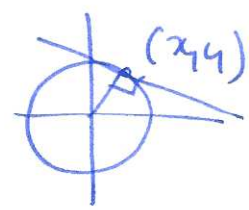
Suppose $y^4 + xy = x^3 - x + 2$

can't solve for y explicitly

$(y(x))^4 + xy(x) = x^3 - x + 2$ ← diff wrt x using chain rule on $y(x)$

$4(y(x))^3 \cdot y'(x) + y(x) + xy'(x) = 3x^2 - 1$

$y'(x)\left(4y^3 + x\right) = 3x^2 - 1 - y$ $\quad y'(x) = \dfrac{3x^2 - 1 - y}{4y^3 + x}$



<u>Example</u> $x^2 + y^2 = 1$ $\quad 2x + 2yy' = 0$ $\quad y' = -x/y$

<u>Application</u>: derivatives of inverse functions.

<u>Example</u> $y = \ln(x)$

$e^y = x$

$e^y \cdot y' = 1$

$y' = e^{-y} = e^{-\ln(x)} = 1/x$

<u>Thm</u>: If $f(x)$ differentiable, one-to-one, with inverse $f^{-1}(x) = g(x)$

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$