<u>Thm</u> $\frac{d}{dx}(x^n) = nx^{n-1}$

<u>Proof</u> $f(x) = x^n \quad f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

binomial theorem : $(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \binom{n}{3}x^{n-3}h^3 + \dots + h^n}_{\text{all of these contain } h^2}$

so $\lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{\cancel{x^n} + nx^{n-1}h + h^2(\text{—}) - \cancel{x^n}}{h} = \lim_{h\to 0} nx^{n-1} + h(\text{—})$

$= nx^{n-1}$ □

<u>warning</u> : this rule works for polynomials only, <u>not</u> exponentials.

$f(x) = x^{100}$ polynomial $\qquad f(x) = 2^x$ <u>not</u> polynomial.

<u>other useful rules</u>

<u>Thm</u> (linearity) If f and g are differentiable functions, then

- $f+g$ is differentiable with $(f+g)' = f' + g'$

$$\leftrightarrow \frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

- k constant, $(kf)' = kf' \leftrightarrow \frac{d}{dx}(kf) = k\frac{df}{dx}$

<u>Proof</u> (follows from limit laws)

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$$

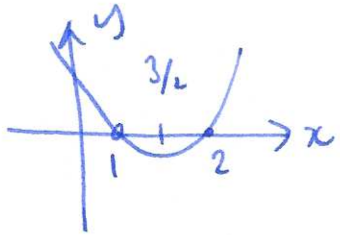
$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$

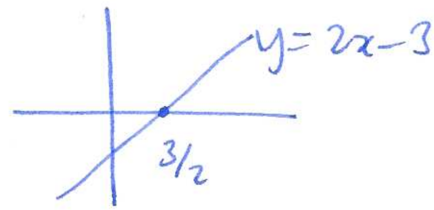
$$(kf)'(x) = \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} = k\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = kf'(x) \; □$$

Example $f(x) = x^2 - 3x + 2$ find $f'(x)$

$$\frac{df}{dx} = \frac{d}{dx}\left(x^2 - 3x + 2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(-3x\right) + \frac{d}{dx}(2)$$
$$= 2x - 3 + 0$$

graphs





Derivative of e^x

consider $f(x) = b^x$, $b > 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} b^x \frac{(b^h - 1)}{h}$$

$$= b^x \underbrace{\lim_{h \to 0} \frac{b^h - 1}{h}}$$ doesn't depend on x! assume this limit exists and call it m_b.

we have shown: for exponential functions the derivative is <u>proportional</u> to the value of the original function, i.e. if $f(x) = b^x$, $f'(x) = m_b b^x$.

in particular, slope at $x = 0$ is m_b.

<u>recall</u>: e is defined to be the <u>special number</u> s.t. the slope of e^x at $x = 0$ is equal to 1. Therefore, if $\boxed{f(x) = e^x, \ f'(x) = e^x}$ $\left(\boxed{\frac{d}{dx}(e^x) = e^x}\right)$

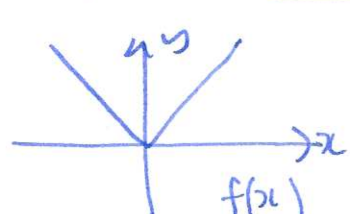
Example $\frac{d}{dx}\left(7e^x + 8x^2\right) = 7e^x + 16x$

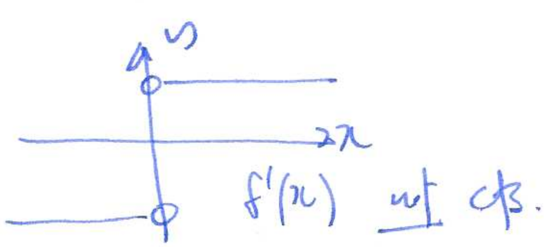
observation this shows that e^x is <u>not</u> a polynomial

$\frac{d^n}{dx^n}\left(p(x)\right) \leftarrow$ degree goes down, eventually zero.

<u>Thm</u> Differentiable $\Rightarrow$ continuous. <u>Warning</u>: continuous $\not\Rightarrow$ differentiable.

Example $f(x) = |x|$
continuous





<u>claim</u>: $f(x) = |x|$ not differentiable at $x=0$

<u>check</u>: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h} \overset{x=0}{=} \lim_{h\to 0} \frac{|h|}{h}$

$\lim_{h\to 0+} \frac{|h|}{h} = \lim_{h\to 0+} 1 = 1$ $\quad \lim_{h\to 0-} \frac{|h|}{h} = \lim_{h\to 0-} -1 = -1$ $\quad +1 \neq -1$

so $\lim_{h\to 0} \frac{|h|}{h}$ DNE. $\square$.

<u>local picture</u>: if $f(x)$ is differentiable at $x=c$, then if you look close enough, the graph looks close to a straight line.

<u>Proof</u> (differentiable $\Rightarrow$ cts) $f(x)$ differentiable at $x=c$ means

$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. (want to show $\lim_{x\to c} f(x) = f(c)$)

consider $f(c+h) - f(c) = h \cdot \frac{(f(c+h) - f(c))}{h}$ so $\lim_{h\to 0} f(c+h) - f(c) =$

$\lim_{h\to 0} h \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0 \cdot f'(c) = 0$ $\square$.

§3.3 Product and quotient rules

new functions from old: $f(x)g(x)$ product $\quad \frac{f(x)}{g(x)}$ quotient

<u>Thm</u> (product rule) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dx}(fg) = \frac{df}{dx} g + f \frac{dg}{dx}$$

<u>Warning</u> $(fg)' \neq f'g'$. !!

<u>Examples</u> ① $\frac{d}{dx}(x^2) = \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x) = x + x = 2x$

② $\frac{d}{dx}(3x^2(x^2+1)) = (3x^2)'(x^2+1) + (3x^2)(x^2+1)'$

$= 6x(x^2+1) + 3x^2 \cdot 2x$

③ $\frac{d}{dx}(x^2 e^x) = \frac{d}{dx}(x^2)e^x + x^2 \frac{d}{dx}(e^x) = 2xe^x + x^2e^x$

Proof (of product rule) (assume f, g both differentiable at x)

$$(fg)'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f(x)g'(x) + f'(x)g(x)$$

$\square$

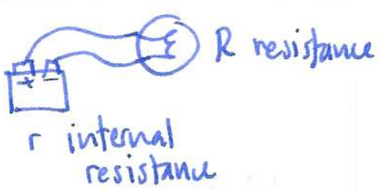
Thm Quotient rule (assume f, g differentiable at x, $g(x) \neq 0$)

then $\left(\frac{f}{g}\right)'(x) = \frac{gf' - fg'}{g^2} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

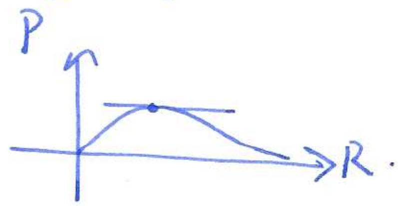
Examples ① $\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$

② $\frac{d}{dt}\left(\frac{e^t}{e^t+t}\right) = \frac{(e^t+t)(e^t)' - (e^t+t)'(e^t)}{(e^t+t)^2} = \frac{(e^t+t)e^t - (e^t+1)e^t}{(e^t+t)^2} = \frac{te^t - e^t}{(e^t+t)^2}$

③ application: battery power



power $P = \frac{V^2R}{(r+R)^2}$



Q: when does the battery give maximal power?

A: when $\frac{dP}{dR} = 0$ $\quad P = \frac{V^2R}{(r+R)^2}$ $\quad$ ($P(R)$, V, r constants)

$$\frac{dP}{dR} = \frac{(r+R)^2(V^2R)' - ((r+R)^2)'V^2R}{(r+R)^4} = \frac{(r+R)^2V^2 - (r^2+2Rr+R^2)'V^2R}{(r+R)^4}.$$

$$= \frac{V^2\left[(r+R)^2 - R(2r+2R)\right]}{(r+R)^4} = \frac{V^2(r+R)(r+R-2R)}{(r+R)^4} = \frac{V^2(r-R)}{(r+R)^3} = 0$$

$\Rightarrow R = r.$