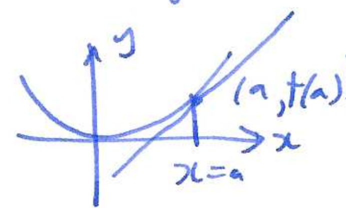
If this limit exists, we say that the function $f(x)$ is <u>differentiable</u> at $x=a$ (19)

<u>note</u>: $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ same as $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$

<u>Defn</u> the tangent line to $f(x)$ at the point $(a, f(a))$ is the straight line with slope $f'(a)$ through $(a, f(a))$



the equation for this line is $y-y_0 = m(x-x_0)$

i.e $y-f(a) = f'(a)(x-a)$ or $y = f(a)+f'(a)(x-a)$

<u>Example</u> find the tangent line to $y=x^2$ at $x=1$

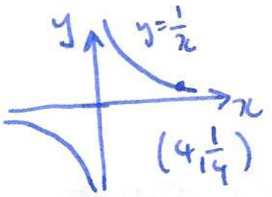
$(x, f(x))$ is $(1,1)$ slope $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0} \dfrac{(1+h)^2-1}{h}$

$= \lim\limits_{h\to 0} \dfrac{1+2h+h^2-1}{h} = \lim\limits_{h\to 0} 2+h = 2$

so equation of tangent line is $y-1 = 2(x-1)$ $\quad y = 1+2(x-1)$

$y = 2x-1$

<u>Example</u> find slope of tangent line to $f(x) = \frac{1}{x}$ at $x=4$



slope $f'(4) = \lim\limits_{h\to 0} \dfrac{f(4+h)-f(4)}{h} = \lim\limits_{h\to 0} \dfrac{1/4+h - 1/4}{h}$

$= \lim\limits_{h\to 0} \dfrac{1}{h} \dfrac{4-(4+h)}{4(4+h)} = \lim\limits_{h\to 0} \dfrac{-h}{h\,4(4+h)} = \lim\limits_{h\to 0} \dfrac{-1}{16+4h} = -\dfrac{1}{16}$

tangent line: $y - \frac{1}{4} = -\frac{1}{16}(x-4)$

<u>Example</u> find slope of straight line $y = mx+b$

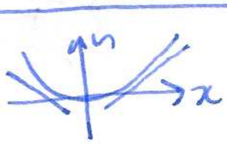
find slope at $x=a$: $f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \dfrac{m(a+h)+b-(ma+b)}{h}$

$= \lim\limits_{h\to 0} \dfrac{ma+mh+b-ma-b}{h} = \lim\limits_{h\to 0} m = m$

<u>observation</u> if $f(x) = b$ (constant), then $f'(x) = 0$ for all x.

<u>§ 3.2 Derivative as a function</u>

at each point x, there is a tangent line, with a slope, which is a number



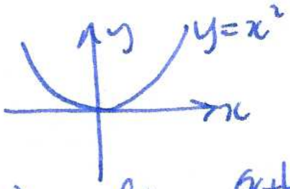
$f: \mathbb{R} \to \mathbb{R}$

$x \mapsto x^2$

Therefore we can define a function $x \mapsto$ slope of tangent line at x

notation: we call this function $f'(x)$ or "the derivative of f"

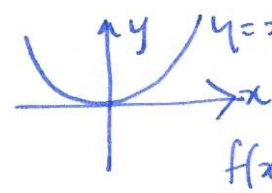
Example

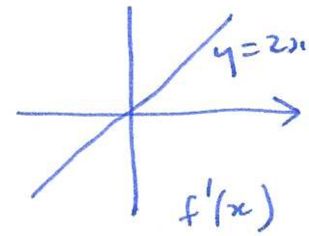


then $\displaystyle f'(x) \overset{\text{slope at } x}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ if $f(x) = x^2$

then $\displaystyle f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} 2x+h = 2x$

summary if $f(x) = x^2$, then $f'(x) = 2x$

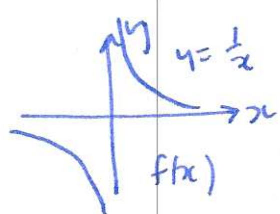


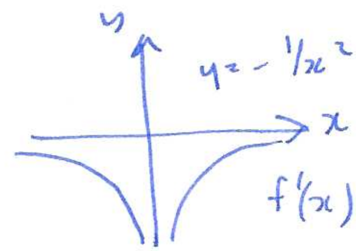


Example $f(x) = \frac{1}{x}$

$\displaystyle f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{x-(x+h)}{(x+h)x} = \lim_{h\to 0} \frac{-h}{h(x+h)x}$

$\displaystyle = \lim_{h\to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$





Remarks ① functions: f: domain $\to$ range, e.g. $f: \mathbb{R} \to \mathbb{R}$

derivative: (differentiable) functions $\to$ functions

$f(x) \longmapsto f'(x)$

warning: not all functions differentiable.

② "calculus" means rules for doing calculations – we don't have to explicitly compute limits all the time.

Example $f(x) = x^3$ $\displaystyle f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}$

$\displaystyle = \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2.$

Thm (powers of x) if $f(x) = x^n$, $f'(x) = nx^{n-1}$ (works for all $n \in \mathbb{R}$!)

Examples $\frac{d}{dx}(x^2) = 2x$ $\quad \frac{d}{dx}(x^3) = 3x^2$ $\quad \frac{d}{dx}(x^1) = 1$ $\quad \frac{d}{dx}(1 = x^0) = 0$

$\frac{d}{dx}\left(x^{-1} = \frac{1}{x}\right) = -\frac{1}{x^2}$ $\quad \frac{d}{dx}\left(x^{1/2} = \sqrt{x}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$