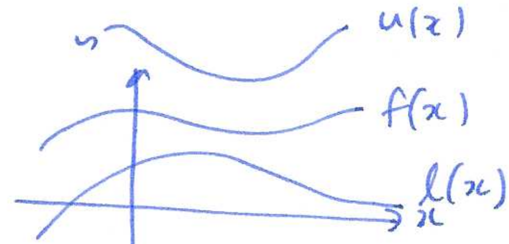
§ 2.6 Trigonometric limits

Q: what is $\lim_{x\to 0} \frac{\sin x}{x}$? substitute $x=0$ get $\frac{0}{0}$ undefined.

A: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (try $x = 0.1, 0.01$, etc...)

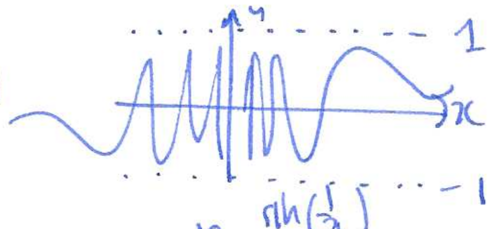


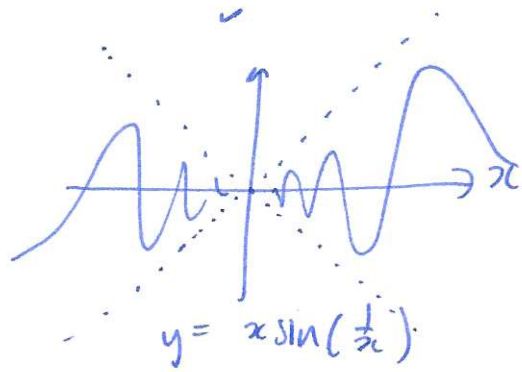
squeeze theorem : suppose $l(x) \leq f(x) \leq u(x)$

Thm suppose $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = L = \lim_{x\to c} u(x)$

then $\lim_{x\to c} f(x) = L$

example $\lim_{x\to 0} x \sin(\frac{1}{x})$





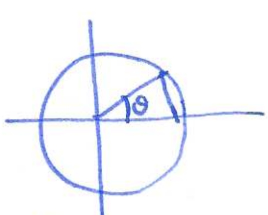
$-1 \leq \sin(x) \leq 1$

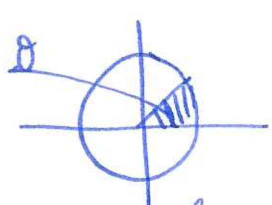
$-|x| \leq x\sin(x) \leq |x|$ $\quad \lim_{x\to 0} |x| = 0 \quad \lim_{x\to x} -|x| = 0$

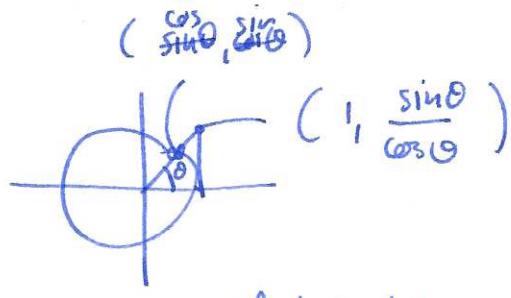
$\Rightarrow \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$

Thm $\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos(x)}{x} = 0$

Proof (of $\lim_{x\to 0} \frac{\sin x}{x} = 1$) assume $0 < \theta < \frac{\pi}{2}$. Consider the following three areas:







area of triangle $\frac{1}{2}bh$ $\qquad$ area of sector $\qquad$ area of triangle

$\frac{1}{2} \cdot 1 \cdot \sin\theta \quad \leq \quad \pi r^2 \frac{\theta}{2\pi} \quad \leq \quad \frac{1}{2} \frac{\sin\theta}{\cos\theta}$

$\frac{1}{2}\sin\theta \quad \leq \quad \frac{1}{2}\theta \quad \leq \quad \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

$\frac{\sin\theta}{\theta} \leq 1 \qquad \cos\theta \leq \frac{\sin\theta}{\theta}$

so $\cos\theta \le \dfrac{\sin\theta}{\theta} \le 1$ $\qquad \lim_{\theta\to 0} 1 = 1 \qquad \lim_{\theta\to 0} \cos\theta = 1$

squeeze theorem $\Rightarrow \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$ $\square$.

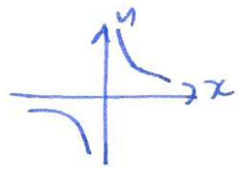
Examples ① $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$ $\qquad$ know: $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

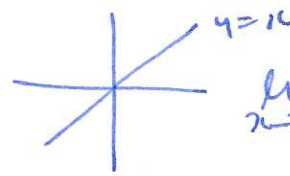
write $3x = \theta$: $\lim_{\theta\to 0} \dfrac{\sin\theta}{2(\theta/3)} = \lim_{\theta\to 0} \dfrac{3}{2} \dfrac{\sin\theta}{\theta} = \dfrac{3}{2} \lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = \dfrac{3}{2}$.

② $\lim_{t\to 0} \dfrac{1-\cos t}{\sin t} = \lim_{t\to 0} \dfrac{1-\cos t}{t} \cdot \dfrac{t}{\sin t} = \underbrace{\lim_{t\to 0} \dfrac{\sin t}{t}}_{1} \cdot \underbrace{\lim_{t\to 0} \dfrac{1-\cos t}{t}}_{0} = 0$

§2.7 Limits at infinity

key observation : $\lim_{x\to\infty} \dfrac{1}{x} = 0$





$\lim_{x\to\infty} x = +\infty$.

Examples

① $\lim_{x\to\infty} \dfrac{3x}{2x-1} = \lim_{x\to\infty} \dfrac{3}{2-1/x} = \dfrac{3}{2}$.

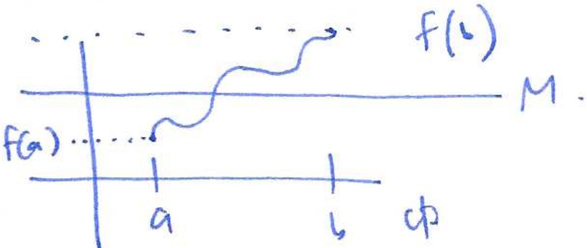
② $\lim_{x\to\infty} \dfrac{x^2+x}{x-3} = \lim_{x\to\infty} \dfrac{x+1}{1-3/x} = +\infty$.

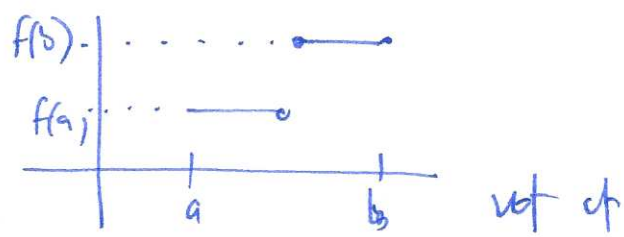
③ $\lim_{x\to\infty} \dfrac{1}{x} - \dfrac{2}{3x+1} = \lim_{x\to\infty} \dfrac{1}{x} - \lim_{x\to\infty} \dfrac{2}{3x+1} = 0 - 0 = 0$

④ $\lim_{x\to 0} \dfrac{\sqrt{2x^2+1}}{4x+1} = \lim_{x\to 0} \dfrac{\sqrt{2+1/x^2}}{4+1/x} = \dfrac{\sqrt{2}}{4}$.

§2.8 Intermediate Value Theorem (IVT)

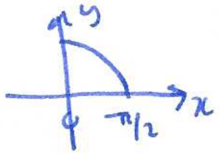
"continuous functions can't skip values"





Thm (Intermediate Value Theorem IVT)
If $f(x)$ is a cts function on a closed interval $[a,b]$ with $f(a) \neq f(b)$, then for any number M between $f(a)$ and $f(b)$ there is at least one $c \in [a,b]$ s.t. $f(c) = M$. $\square$.

Example show $\cos(x) = \frac{1}{4}$ has at least one solution



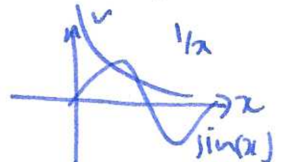
consider $[0, \frac{\pi}{2})$ $\left.\begin{array}{l}\cos(0) = 1 \\ \cos(\pi/2) = 0\end{array}\right\}$ $0 \leq \frac{1}{4} \leq 1 \Rightarrow \exists c$ with $\cos(c) = \frac{1}{4}$.

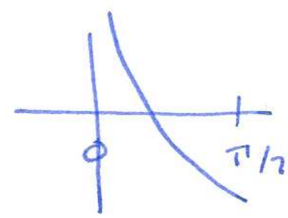
special case: finding zeros

Corollary if $f(x)$ is cts on $[a,b]$ and $f(a), f(b)$ have different sign, then there is at least one $c \in [a,b]$ with $f(c) = 0$.

Bisection method: find a solution to $\sin(x) = \frac{1}{x}$ in $[0, \frac{\pi}{2}]$

consider $f(x) = \frac{1}{x} - \sin(x)$



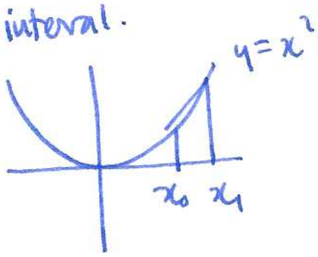


$\left.\begin{array}{l} f(0) = +\infty > 0 \\ f(\frac{\pi}{2}) = \frac{2}{\pi} - 1 < 0 \end{array}\right\}$ midpoint: $\frac{\pi}{4}$ $\quad f(\pi/4) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \approx 0.566 > 0$

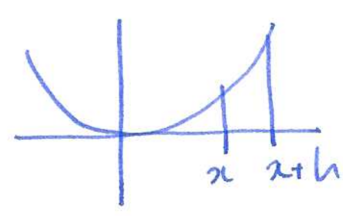
so continue with $[\frac{\pi}{4}, \frac{\pi}{2}]$, etc.

§3.1 Defn of the derivative

Recall: we can compute the average rate of change of a function over an interval.



$[x_0, x_1]$ $\quad \frac{f(x_1) - f(x_0)}{x_1 - x_0}$



Q: how do we compute the slope of the tangent line?
A: look at average rate of change over small interval $[x, x+h]$ and take limit as $h \to 0$.

Defn the slope of the tangent line at $x = a$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Notation: also called the derivative written $f'(a)$ (Newton) or $\frac{df}{dx}(a)$ (Leibniz)