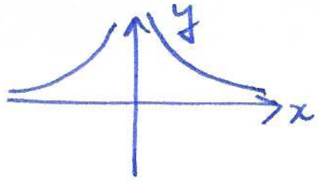
Example $f(x) = \frac{1}{x^2}$



$\lim_{x\to 0+} \frac{1}{x^2} = +\infty$
$\lim_{x\to 0-} \frac{1}{x^2} = +\infty$
$\Bigg\}$ $\lim_{x\to 0} \frac{1}{x^2} = +\infty.$

§2.3 Basic limit laws

Example $\lim_{x\to 0} 2x+2 = \lim_{x\to 0} 2x + \lim_{x\to 0} 2 = 2\lim_{x\to 0} x + 2 = 2.$

Thm assume that $\lim_{x\to c} f(x)$, $\lim_{x\to c} g(x)$ exist and are finite. Then:

1) sums: $\lim_{x\to c} (f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

2) constant multiple: $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$ k constant (does not depend on x)

3) products: $\lim_{x\to c} (f(x)\,g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$

4) quotients: $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ as long as $\lim_{x\to c} g(x) \neq 0$

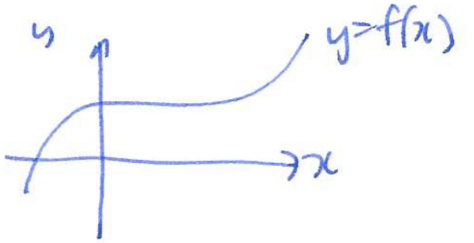
Warning: these rules don't work if either $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ DNE.

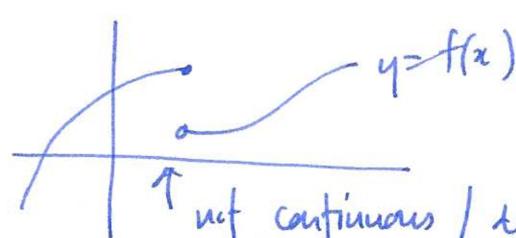
Examples: $\lim_{x\to 3} x^2 = \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} x\right) = 3.3 = 9$

$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\lim_{t\to 2} t+5}{\lim_{t\to 2} 3t} = \frac{7}{6}$

§2.4 Limits and continuity

Example

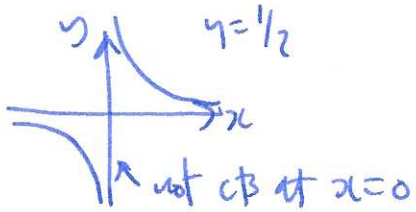


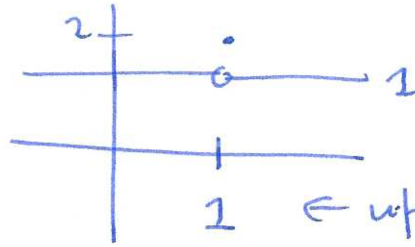


not continuous / discontinuous

<u>Defn</u> we say $f(x)$ is <u>continuous</u> at $x=c$ if $\lim_{x\to c} f(x) = f(c)$.
If the limit does not exist / is infinite | does not equal $f(c)$, then $f(x)$ is <u>not</u> continuous at $x=c$.

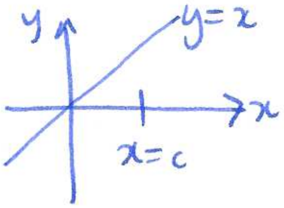
<u>Examples</u>.





$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1 \end{cases}$

<u>Example</u> show $f(x) = x$ is cts



$f(c) = c$ want to show $\lim_{x\to c} f(x) = f(c)$

follows from limit laws: $\lim_{x\to c} x = c$ ✓

<u>Defn</u> $f(x)$ is <u>left continuous</u> at $x=c$ if $\lim_{x\to c^-} f(x) = f(c)$
<u>right continuous</u> at $x=c$ if $\lim_{x\to c^+} f(x) = f(c)$

If at least one of the left or right limits is $\pm\infty$ we say $f(x)$ has an infinite discontinuity at $x=c$

<u>Building continuous functions</u>

<u>Thm</u> 0 $f(x) = k$, $f(x) = x$ are continuous.

<u>Thm</u> 1 Suppose $f(x)$ and $g(x)$ are both continuous at $x=c$, then the following functions are cts at $x=c$:

1) $f(x) + g(x)$ 2) $kf(x)$ for any constant k

3) $f(x)g(x)$ 4) $\dfrac{f(x)}{g(x)}$ if $g(c) \neq 0$

<u>Proof</u>: these follow directly from the limit laws

check 1) $f(x)$ cts means $\lim_{x\to c} f(x) = f(c)$

$g(x)$ cts means $\lim_{x\to c} g(x) = g(c)$

so $\lim\limits_{x\to c} (f(x)+g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x\to c} g(x) = f(c)+g(c)$ as required. $\square$

Thm 2 Polynomials are continuous $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$
Rational functions $\dfrac{p(x)}{q(x)}$ are cts, except where $q(x)=0$

Proof $f(x) = x$ is cts
so $f(x) \cdot f(x) = x \cdot x = x^2$ is continuous (product)
similarly, x^n is continuous
so $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is cts (multiply by constant, addition)
so $\dfrac{p(x)}{q(x)}$ is cts (quotients) where $q(x) \neq 0$. $\square$.

Useful facts

Thm 3
- $\sin(x)$, $\cos(x)$ are continuous
- b^x is cts
- $\log_b(x)$ is cts
- $x^{1/n}$ is cts

(combinations of these with polynomials are sometimes called elementary functions)

Thm 4 (inverse functions) If $f: D \to \mathbb{R}$ is cts, with inverse $f^{-1}: \mathbb{R} \to D$, then f^{-1} is cts.

Thm 5 (composition) If $f(x)$ is cts at $x=c$, and $g(x)$ is cts at $x = f(c)$, then $g(f(x))$ is cts at $x=c$

$c \xrightarrow{f} f(c) \xrightarrow{g} g(f(c))$ Example $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x+1}}$ cts at $x=1$

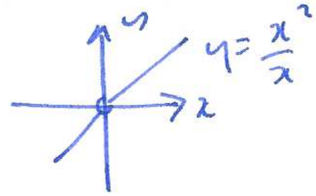
Q: where is $f(x) = \frac{x^2}{\sin(x)}$ cts?

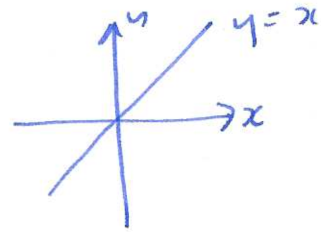
§2.5 Evaluating limits algebraically

Example $\frac{x^2}{x}$ undefined at $x=0$: $\frac{0}{0}$ indeterminate form

but $\lim_{x\to 0} \frac{x^2}{x}$ does not depend on value at $x=0$

$\frac{x^2}{x} = x$ for $x \neq 0$





so $\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$

Indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 0^0

note $\frac{1}{0}$ not indeterminate, gives limit $\pm\infty$ or DNE.

Examples $\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12}$ $\quad x=3$: $\frac{9-12+3}{9+3-12} = \frac{0}{0}$

factor : $\frac{(x-3)(x+1)}{(x-3)(x+4)} = \frac{x-1}{x+4}$ $(x \neq 3)$

so $\lim_{x\to 3} \frac{x^2-4x+3}{x^2+x-12} = \lim_{x\to 3} \frac{x-1}{x+4} = \frac{2}{7}$

Example $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ $\quad \frac{2-2}{4-4} = \frac{0}{0}$

$\frac{\sqrt{x}-2}{x-4} = \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$ $(x \neq 4)$ so $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

Example $\lim_{x\to \pi/2} \frac{\tan x}{\sec x}$ $\quad \lim_{x\to 1} \underbrace{\frac{1}{x-1} - \frac{2}{x^2-1}}$

$$\lim_{x\to \pi/2} \frac{\sin x / \cos x}{1/\cos x} = \lim_{x\to \pi/2} \sin x = 1$$

$$= \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1} = \frac{x-1}{(x+1)(x-1)} = \frac{1}{x+1} \ (x \neq 1)$$

$$\lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}.$$