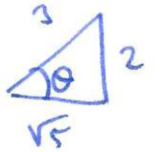
special case (shift): $\sin\left(x+\frac{\pi}{2}\right) = \cos(x)$

<u>Example</u> suppose $\sin\theta = \frac{2}{3}$ find $\cos\theta$, $\tan\theta$, $\sin 2\theta$



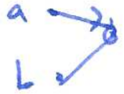
$\cos\theta = \frac{\sqrt{5}}{3}$ $\tan\theta = \frac{2}{\sqrt{5}}$ $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$

§1.5 Inverse functions

recall: $f: X \to Y$
domain range
$x \mapsto f(x)$

want: the inverse function should be the reverse of this
$X \leftarrow Y : f^{-1}$
$x \leftarrow f(x)$

<u>problem</u>: the inverse is often not a function



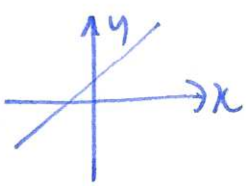
suppose $a \neq b$ but $f(a) = f(b)$, what is $f^{-1}\left(\begin{matrix} f(a) \\ f(b) \end{matrix}\right)$?

Q: when does a function have an inverse?

A: when it passes the <u>horizontal line test</u> (one-to-one | injective)

$\Leftrightarrow$ for each number $c \in$ range, there is a unique x s.t. $f(x) = c$.

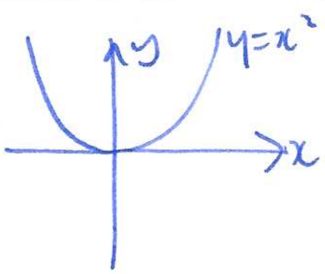
<u>Example</u> $y = x+1$



Q: how do we find a formula for the inverse?

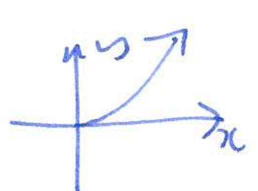
<u>A</u>:
1. write down $y = f(x)$
2. solve for x in terms of y, i.e. $x = g(y)$
3. $f^{-1}(x) = g(x)$
4. check!

<u>Bad example</u> $f(x) = x^2$



<u>problem</u>: doesn't pass horizontal line test

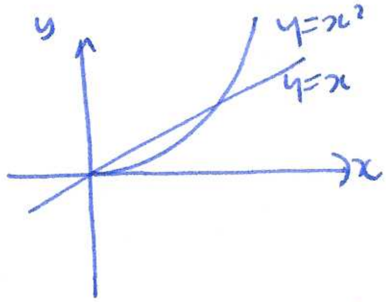
<u>fix</u>: restrict domain, consider $f: [0,\infty) \to [0,\infty)$
$x \mapsto x^2$



<u>does</u> pass horizontal line test
so has an inverse we call $\sqrt{x}$

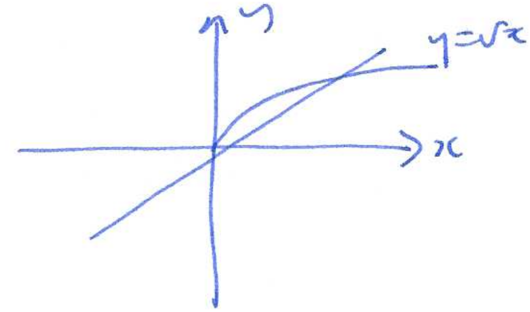
$f^{-1}: [0,\infty) \to [0,\infty)$
$x \mapsto \sqrt{x}$.

How to draw the graph of the inverse



- reflect in $y=x$

<u>reason</u>:
graph of f: $(x, f(x))$
f^{-1}: $(x, f^{-1}(x))$

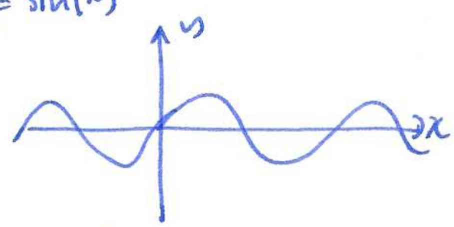


$y = f(x) \Leftrightarrow f^{-1}(y) = x \qquad (x, f(x)) \leftrightarrow (f^{-1}(y), y)$.

$\leftrightarrow$ swap and relabel.

Inverse trig functions

$y = \sin(x)$

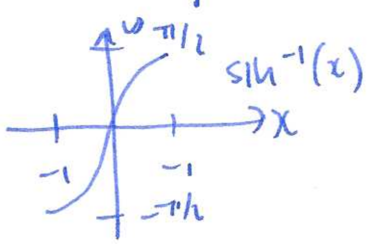


<u>problem</u>: not one-to-one

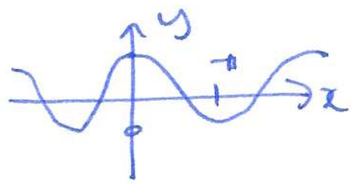
<u>fix</u>: restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

$\sin^{-1}(x) : [-1, 1] \mapsto [-\frac{\pi}{2}, \frac{\pi}{2}]$



<u>similarly</u> $\quad y = \cos(x)$



restrict domain to $[0, \pi]$

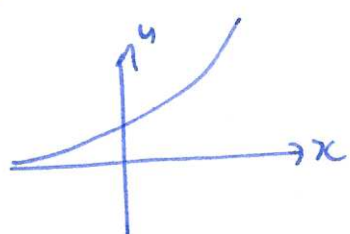
$\cos(x) : [0, \pi] \to [-1, 1]$

$\cos^{-1}(x) : [-1, 1] \to [0, \pi]$

§1.6 Exponential and logarithm functions

<u>Example</u> $\quad x \mapsto 2^x$

x	-2	-1	0	1	2	3
2^x	$1/4$	$1/2$	1	2	4	8



can use any positive number instead of 2: $f(x) = b^x$
$(b > 0)$

useful properties:

- positive $b^x > 0$
- b^x increasing if $b > 1$
- decreasing if $b < 1$
- b^x grows faster than any polynomial

<u>exponent rules</u> : $b^0=1$ $\quad b^x b^y = b^{x+y}$ $\quad b^{-x} = \frac{1}{b^x}$ $\quad \frac{b^x}{b^y} = b^{x-y}$

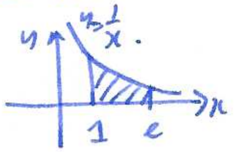
$(b^x)^y = b^{xy}$ $\quad b^{1/n} = \sqrt[n]{b}$

• there is a special exponential function e^x, $e = 2.71828\ldots$

<u>key properties</u>

① e is the unique number such that e^x has slope 1 at $x=0$

② e is the unique number s.t. the area under the curve $y=\frac{1}{x}$ between 1 and e has area 1



<u>logarithms</u> The logarithm is the <u>inverse function</u> for the exponential function

the special logarithm with base $b=e$ is called the <u>natural logarithm</u> $\ln(x)$

• inverse function properties: $f^{-1}(f(x)) = x = f(f^{-1}(x))$

$$b^{\log_b(x)} = x = \log_b(b^x)$$

• logarithm rules: $\log_b(1) = 0$ $\quad \log_b(b) = 1$

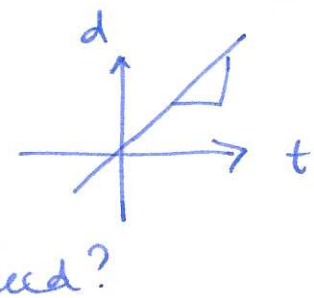
$\log_b(st) = \log_b(s) + \log_b(t)$ $\quad \log_b\left(\frac{1}{t}\right) = -\log_b(t)$ $\quad \log_b\left(\frac{s}{t}\right) = \log_b(s) - \log_b(t)$

$\log_b(s^t) = t\log_b(s)$

convert between bases: $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ for any a, so $= \frac{\ln(x)}{\ln(b)}$.

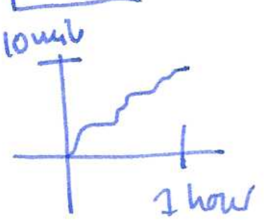
<u>§2.1 Limits rates of change, tangent lines</u>

<u>motivation</u> : velocity $\quad$ example: driving at constant speed



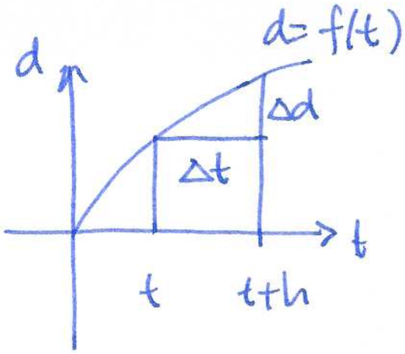
velocity $= \frac{\text{distance}}{\text{time}} =$ slope of line

<u>problem</u> : what happens if you don't travel at constant speed?

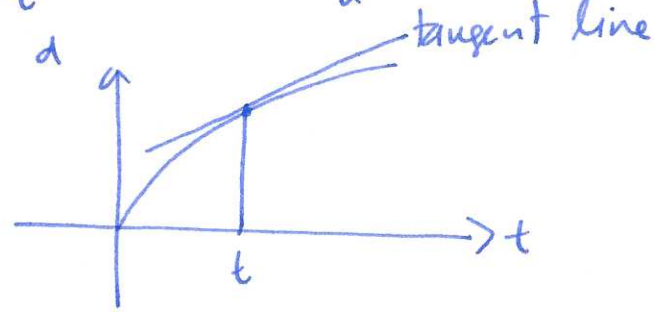


average speed $= \frac{\text{distance travelled}}{\text{time taken}}$

we can look at average speed over any time interval, including very short ones.



average speed on an interval $[t, t+h]$ is $\frac{\Delta d}{\Delta t} = \frac{f(t+h) - f(t)}{t+h-t} = \frac{f(t+h)-f(t)}{h}$



<u>Q</u>: what is the speed at time t?
(sometimes called the instantaneous speed / rate of change)

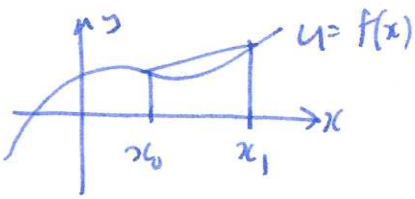
<u>A</u>: speed is the slope of the tangent line at t

<u>idea/hope</u>: as the length of the interval $[t, t+h]$ gets small, the average speed gets close to the slope of the tangent line.
This works for "nice" functions.

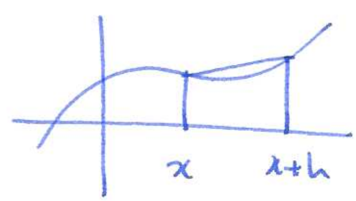
<u>observation</u>: this works for any function $y = f(x)$, not just speed.

<u>summary</u>: average rate of change over an interval $[x_0, x_1]$ is

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



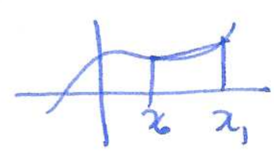
often:



$$\frac{f(x+h) - f(x)}{h}.$$

<u>§2.2 Limits</u> aim: want to find slope of tangent lines.

<u>know</u>: average rate of change $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$



<u>Q</u>: why can't we just set $x_1 = x_0$?

<u>A</u>: doesn't work, get $\frac{f(x_0)-f(x_0)}{x_0 - x_0} = \frac{0}{0}$ undefined.

<u>observations</u>

① if we draw careful pictures, the average slope gets close to the slope of the tangent line as the length of the interval gets small.

② seems to work for sample calculations: