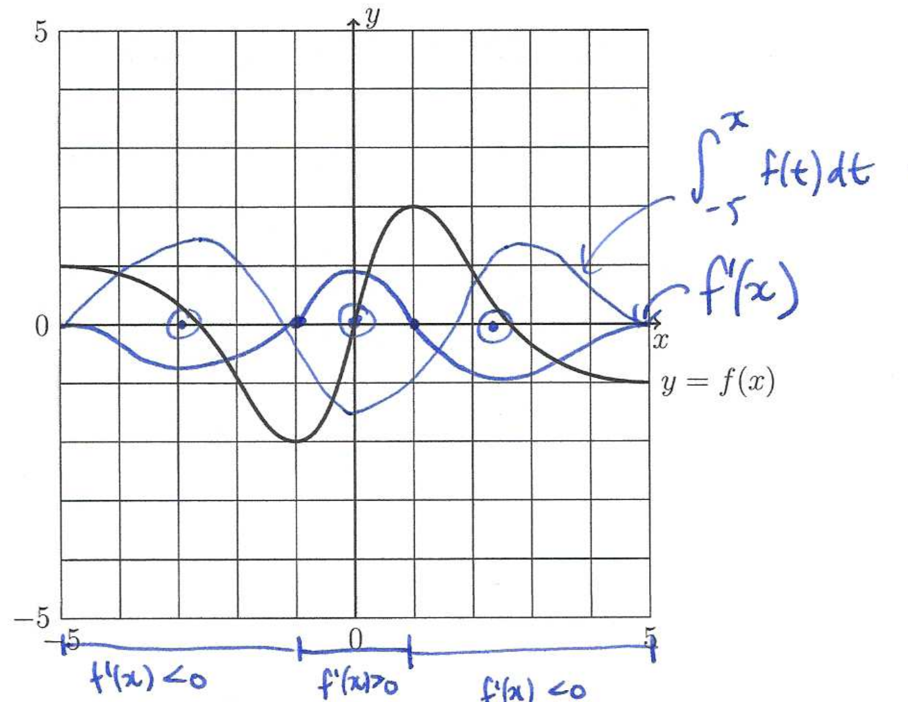
Solutions

Math 231 Calculus 1 Fall 21 Sample Midterm 3

(1) Consider the function $f(x)$ defined by the following graph.



(a) Label all regions where $f'(x) < 0$.
(b) Label all regions where $f'(x) > 0$.
(c) What is $\lim_{x\to\infty} f(x)$? -1
(d) What is $\lim_{x\to-\infty} f'(x)$? 0
(e) What is $\lim_{x\to\infty} f''(x)$? 0
(f) Sketch a graph of $f'(x)$ on the figure.
(g) Sketch a graph of $\int_{-5}^{x} f(t)dt$ on the figure.
(h) Label the approximate locations of all points of inflection. $\odot$

Q2 $f(x) = e^{4-x^2}$

a) no vertical asymptotes. $\lim_{x\to\pm\infty} 4-x^2 = -\infty \Rightarrow \lim_{x\to\pm\infty} f(x) = 0$

b) $f'(x) = e^{4-x^2}\cdot(-2x)$ $\quad f'(x)=0 \Rightarrow x=0 \leftarrow$ critical point.

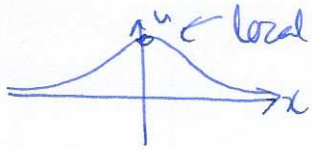
c) $f'(x)<0$ when $x>0$ $\quad f'(x)>0$ when $x<0$

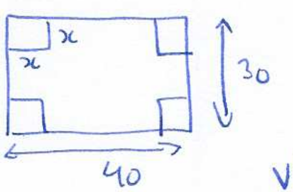
d) $f''(x) = e^{4-x^2}\cdot 4x^2 + e^{4-x^2}\cdot(-2)$ $\quad$ solve $f''(x)=0$: $e^{4-x^2}(4x^2-2)=0$

$\Rightarrow x = \pm 1/\sqrt{2}$.

e) $f''(x)>0$ on $(-\infty, -1/\sqrt{2})$ and $(1/\sqrt{2}, \infty)$ $\quad f''(x)<0$ on $(-1/\sqrt{2}, 1/\sqrt{2})$.

f) $f''(0) = e^4\cdot -2 < 0 \Rightarrow$ local max.

g) 

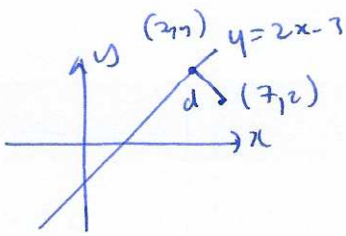
Q3 

$V = x(30-2x)(40-2x) = 4x(15-x)(20-x)$

$= 4x(x^2-35x+300) = 4(x^3-35x^2+300x)$.

$V'(x) = 4(3x^2-70x+300)$

critical points $V'(x)=0$: $x = \dfrac{70 \pm\sqrt{70^2-4\cdot3\cdot300}}{6} \approx 5.657\ldots$

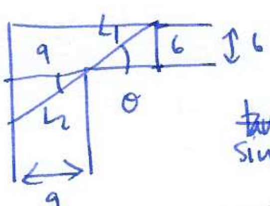
Q4 

$d^2 = (7-x)^2+(2-y)^2 = (7-x)^2+(2-(2x-3))^2$

$= (7-x)^2+(5-2x)^2 = 49-14x+x^2$
$+25-10x+4x^2$

$d^2 = 5x^2-24x+74$

$\dfrac{d(d^2)}{dx} = 10x-24$ $\quad x = \dfrac{24}{10} = \dfrac{12}{5}$, $y = \dfrac{24}{5}-3 = \dfrac{9}{5}$.

Q5 

$L = L_1+L_2 = \dfrac{6}{\sin\theta} + \dfrac{9}{\cos\theta}$ $\quad \dfrac{dL}{d\theta} = -6(\sin\theta)^{-2}\cdot\cos\theta - 9(\cos\theta)^{-2}\cdot(-\sin\theta)$

$\cancel{\tan\theta}$ $\sin = \dfrac{6}{L_1} =$ $\quad = \dfrac{-6\cos\theta}{\sin^2\theta} + \dfrac{9\sin\theta}{\cos^2\theta}$

$\cos\theta = \dfrac{9}{L_2}$

solve $\dfrac{dL}{d\theta}=0$: $\dfrac{6\cos\theta}{\sin^2\theta} = \dfrac{9\sin\theta}{\cos^2\theta}$ $\quad 6\cos^3\theta = 9\sin^3\theta$ $\quad \tan^3\theta = \dfrac{6}{9} = \dfrac{2}{3}$. $\quad \theta_0 = \tan^{-1}(\sqrt[3]{2/3})$

$L = 6/\sin\theta_0 + 9/\cos\theta_0 \approx 21.07$

Q6 a) $\lim\limits_{x\to 4} \dfrac{3x^2-11x-4}{2x^2-3x-20} = \lim\limits_{x\to 4} \dfrac{6x-11}{4x-3} = \dfrac{24-11}{16-3} = \dfrac{13}{13} = 1$

b) $\lim\limits_{x\to 0} \dfrac{\tan 3x}{2x} = \lim\limits_{x\to 0} \dfrac{\sec^2(3x)\cdot 3}{2} = \dfrac{3}{2}$

c) $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9} = \lim\limits_{x\to 9} \dfrac{\frac{1}{2}x^{-1/2}}{1} = \dfrac{1}{6}$

d) $\lim\limits_{x\to 8} \dfrac{2x^2-31x+20}{\sqrt{x+1}-3} = \lim\limits_{x\to 8} \dfrac{4x-31}{\frac{1}{2}(x+1)^{-1/2}} = \dfrac{32-31}{\frac{1}{2}\,\frac{1}{3}} = 6$

e) $\lim\limits_{x\to 5} \dfrac{3x^2-75}{1/5 - 1/x} = \lim\limits_{x\to 5} \dfrac{3x^3-75x}{x/5-1} = \lim\limits_{x\to 5} \dfrac{9x^2-75}{1/5} = 750.$

f) $\lim\limits_{x\to 0} \dfrac{\tan^{-1}(8x)}{\sin^{-1}(2x)} = \lim\limits_{x\to 0} \dfrac{1/(1+(8x)^2)\cdot 8}{1/\sqrt{1-(2x)^2}\cdot 2} = \lim\limits_{x\to 0} \dfrac{4\sqrt{1-4x^2}}{1+64x^2} = 4$

g) $\lim\limits_{x\to 0} \dfrac{x^2 e^{x/2}}{\tan^2(3x/4)} = \lim\limits_{x\to 0} \dfrac{2xe^{x/2} + x^2\cdot e^{x/2}\cdot\frac{1}{2}}{2\tan(3x/4)\sec^2(3x/4)\cdot\frac{3}{4}} = \lim\limits_{x\to 0} \dfrac{e^{x/2}}{\frac{3}{2}\sec^2(3x/4)} \cdot \dfrac{2x+\frac{1}{2}x^2}{\tan(3x/4)}.$ $(*)$

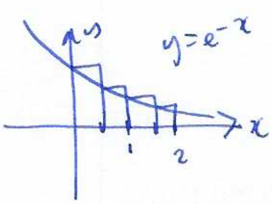
$\lim\limits_{x\to 0} \dfrac{2x+\frac{1}{2}x^2}{\tan(3x/4)} = \lim\limits_{x\to 0} \dfrac{2+x}{\sec^2(3x/4)\cdot 3/4} = \dfrac{8}{3}.$ So $(*) = \dfrac{1}{3/2}\cdot\dfrac{8}{3} = \dfrac{16}{9}$

h) $\lim\limits_{x\to\infty} \dfrac{4x^3+6x^2-9x+1}{2x^3-5x^2+8} = 2.$

i) $\lim\limits_{x\to\infty} \dfrac{e^{x^2}}{x^2+1} = \lim\limits_{x\to\infty} \dfrac{e^{x^2}\cdot 2x}{2x} = \lim\limits_{x\to\infty} e^{x^2} = \infty.$

j) $\lim\limits_{x\to\frac{1}{2}^-} \dfrac{\tan(\pi x)}{\ln(1-2x)} = \lim\limits_{x\to\frac{1}{2}^-} \dfrac{\sec^2(\pi x)\cdot\pi}{(1/(1-2x))(-2)} = \lim\limits_{x\to\frac{1}{2}^-} \dfrac{2\pi(2x-1)}{\cos^2(\pi x)} = \lim\limits_{x\to\frac{1}{2}^-} \dfrac{4\pi}{2\cos(\pi x)\cdot(-\sin(\pi x))\cdot\pi}$ DNE.

Q7.



over estimate

area $\approx \frac{1}{2}\left(e^0 + e^{-1/2} + e^{-1} + e^{-3/2}\right) \approx 1.0988$

Q8 a) $\int x^{-1/2} + 3x^{1/2} - 2x^{3/2}\,dx = 2x^{1/2} + \dfrac{3x^{3/2}}{3/2} - \dfrac{2x^{5/2}}{5/2} + c$

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b) $\int_{-2}^{2} |x| \, dx = \int_{-2}^{0} -x \, dx + \int_{0}^{2} x \, dx = \left[-\frac{1}{2}x^2\right]_{-2}^{0} + \left[\frac{1}{2}x^2\right]_{0}^{2}$

$= 2 + 2 = 4.$

c) $\int_{1}^{27} 2x^{-1/3} \, dx = \left[3x^{2/3}\right]_{1}^{27} = 27 - 3 = 24$

d) $\int_{0}^{2} e^{-4x} \, dx = \left[-\frac{1}{4}e^{-4x}\right]_{0}^{2} = -\frac{1}{4}e^{-8} + \frac{1}{4}$

e) $\int_{0}^{x} \frac{1}{t+2} \, dx = \left[\ln|t+2|\right]_{0}^{x} = \ln(x+2) - \ln(2).$

f) $\int \frac{1}{1+4x^2} \, dx$ $\quad 4x^2 = u^2 \quad u = 2x \quad \frac{du}{dx} = 2 \quad \int \frac{1}{1+u^2} \frac{dx}{du} \, du = \frac{1}{2} \int \frac{1}{1+u^2} \, du = \frac{1}{2}\tan^{-1}(u) + c$

$= \frac{1}{2}\tan^{-1}(2x) + c$

g) $\int \frac{x}{1+4x^2} \, dx \quad u = 1+4x^2 \quad \frac{du}{dx} = 8x \quad \int \frac{x}{u} \cdot \frac{1}{8x} \, du = \frac{1}{8} \int \frac{1}{u} \, du = \frac{1}{8} \ln|u| + c$

$= \frac{1}{8} \ln|1+4x^2| + c$

h) $\int \sin(4x) \, dx \quad u = 4x \quad \frac{du}{dx} = 4 \quad \int \sin(u) \frac{dx}{du} \, du = \frac{1}{4} \int \sin(u) \, du = -\frac{1}{4}\cos(u) + c$

$= -\frac{1}{4}\cos(4x) + c$

i) $\int x \cos(1+x^2) \, dx \quad u = 1+x^2 \quad \frac{du}{dx} = 2x \quad \int x \cos(u) \frac{dx}{du} \, du = \int x \cos(u) \cdot \frac{1}{2x} \, du$

$= \frac{1}{2} \int \cos(u) \, du = \frac{1}{2}\sin(u) + c = \frac{1}{2}\sin(1+x^2) + c$

j) $\int \frac{\sin(x)}{e^{\cos(x)}} \, dx \quad u = -\cos(x) \quad \frac{du}{dx} = +\sin x \quad \int \sin x \cdot e^{+u} \frac{dx}{du} \, du = \int \sin x \, e^u \cdot \frac{1}{\sin x} \, du$

$= \int e^u \, du = e^u + c = e^{-\cos(x)} + c.$

k) $\int \frac{\cos(x)}{\sin^2(x)} \, dx \quad u = \sin x \quad \frac{du}{dx} = \cos x \quad \int \frac{\cos x}{u^2} \frac{1}{\cos x} \, du = \int u^{-2} \, du = -\frac{1}{3}u^{-3} + c$

$= -\frac{1}{3}\sin^{-3}x + c = -\frac{1}{3}\operatorname{cosec}^3 x + c$

Q9

$v(t) = x'(t) = (t+2)^{-5}$

$x(t) = \frac{(t+2)^{-6}}{-6} + x_0$ $\quad x(0) = 0 \Rightarrow \frac{2^{-6}}{-6} + x_0 = 0 \quad x_0 = +\frac{1}{2^5 \cdot 3}$.

$x(t) = \frac{1}{2^5 \cdot 3} - \frac{1}{6 \cdot (t+2)^6}$ $\quad \lim_{t\to\infty} x(t) = \frac{1}{2^5 \cdot 3}$, no doesn't reach $x = 1$.