MTH 231 SMT2 Solutions

Q1 a) $[-1,1]$ b) $(-5,-1)\cup(1,5)$ c) 

d) 1 e) 0 f) $-2,0,2$

Q2,3 a) $f(x) = x^4 e^{-2x^4}$ $\quad f'(x) = 4x^3 e^{-2x^4} + x^4 e^{-2x^4} \cdot -8x^3$

$f'(x) = 4x^3 e^{-2x^4} - 8x^7 e^{-2x^4}$

$f''(x) = 12x^2 e^{-2x^4} + 4x^3 e^{-2x^4} \cdot -8x^3 - 56x^6 e^{-2x^4} - 8x^7 e^{-2x^4} \cdot -8x^3$

b) $f(x) = \dfrac{(1-2x)^{1/2}}{4-\sin(3x)}$ $\quad f'(x) = \dfrac{(4-\sin(3x))\frac{1}{2}(1-2x)^{-1/2}\cdot(-2) - (1-2x)^{1/2}(-\cos(3x)\cdot 3)}{(4-\sin(3x))^2}$

$f''(x) \leftarrow$ skip this too long

c) $f(x) = e^{\ln(x)\cdot x^2}$ $\quad f'(x) = e^{\ln(x)\cdot x^2} \cdot \left(\frac{1}{x}\cdot x^2 + \ln(x)\cdot 2x\right)$

$f'(x) = e^{\ln(x)\cdot x^2}\left(x + 2x\ln(x)\right).$

$f''(x) = e^{\ln(x)\cdot x^2}\left(x+2x\ln(x)\right)^2 + e^{\ln(x)\cdot x^2}\left(1 + 2\ln(x) + 2x\cdot\frac{1}{x}\right)$

d) $f(x) = (\operatorname{cosec}(\ln(x)))^{1/2}$ $\quad f'(x) = \frac{1}{2}(\operatorname{cosec}(\ln(x)))^{-1/2} \cdot -\operatorname{cosec}(\ln(x))\cot(\ln(x)) \cdot \frac{1}{x}$

~~d)~~ $f''(x) \leftarrow$ skip this too long

e) $f(x) = \tan^{-1}\left(\frac{3}{x^{1/3}}\right) = \tan^{-1}\left(3x^{-1/3}\right)$ $\quad f'(x) = \dfrac{1}{1+9x^{-2/3}} \cdot -x^{-4/3}.$

$f''(x) = \dfrac{(1+9x^{-2/3})\cdot\left(\frac{4}{3}x^{-7/3}\right) + 6x^{-5/3}\cdot -x^{-4/3}}{(1+9x^{-2/3})^2}.$

f) $f(x) = \sin^{-1}(3x-2)$ $\quad f'(x) = \dfrac{1}{\sqrt{1-(3x-2)^2}} \cdot 3 = 3\left(1-(3x-2)^2\right)^{-1/2}$

$f''(x) = -\frac{3}{2}\left(1-(3x-2)^2\right)^{-3/2} \cdot 2(3x-2)\cdot 3$

Q4 a) $h(x) = f(x)g(x)$ $\quad h'(x) = f'(x)g(x) + f(x)g'(x)$ $\quad h'(2) = f'(2)g(2) + f(2)g'(2)$

$h'(2) = 1\cdot\frac{1}{2} + 0\cdot -\frac{1}{2} = \frac{1}{2}$

b) $h(x) = f(g(x))$ $\quad h'(x) = f'(g(x))\cdot g'(x)$ $\quad h'(-2) = f'(g(-2))\cdot g'(-2) = f'\left(\frac{1}{2}\right)\cdot\frac{1}{2} = -\frac{1}{2}$

Q5 $4x^2 - y^2 = 4$ $\quad$ at $(2,-2)$: $\quad 16 + 4y' = 0$ $\quad y+2 = -\frac{1}{4}(x-2)$

$8x - 2yy' = 0$ $\quad y' = -\frac{1}{4}$

Q6 $x^2y - x^2y^3 = \cos(xy)$

$$2xy + x^2y' - 2xy^3 - x^2 \cdot 3y^2y' = -\sin(xy)\cdot(y + xy')$$

$$y'(x^2 - 3x^2y^2 + \sin(xy)x) = -\sin(xy)y - 2xy + 2xy^3$$

$$y' = \frac{-\sin(xy)\cdot y - 2xy + 2xy^3}{x^2 - 3x^2y^2 + \sin(xy)x}$$

Q7 $V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = 10$

$A = 4\pi r^2$

$$\left.\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}\right\} \quad 10 = 4\pi \cdot 20^2 \frac{dr}{dt}$$

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} \qquad \text{so } \frac{dr}{dt} = \frac{10}{4\pi \cdot 400} = \frac{1}{160\pi}$$

$$\frac{dA}{dt} = 8\pi \cdot 20 \cdot \frac{1}{160\pi} = 1 \text{ cm}^2/\text{sec}.$$

Q8 $f(x) = \sqrt[3]{x} = x^{1/3}$ $\qquad f(64) = 4$

$f'(x) = \frac{1}{3}x^{-2/3}$. $\qquad f'(64) = \frac{1}{3}\frac{1}{4^2} = \frac{1}{3\times 16} = \frac{1}{48}$

$f(63) \approx f(64) + f'(64)\cdot(-1) = 4 - \frac{1}{48} \approx$ or ~~3.98148~~ 3.979167

percentage error: $\left|\frac{\sqrt[3]{63} - 3.98148}{\sqrt[3]{63}}\right| \times 100 \approx$ ~~0.0618%~~ 0.00275

Q9 $f(x) = e^x(x^2 - x - 7)$ $\qquad f'(x) = e^x(x^2 - x - 7) + e^x(2x - 1)$

$f'(x) = e^x(x^2 + x - 8)$ $\quad x = \frac{-1 \pm \sqrt{1 + 4.8}}{2} = \frac{-1 \pm \sqrt{33}}{2}$

e^x	+	+	+
$x + \frac{1-\sqrt{33}}{2}$	−	+	+
$x + \frac{1+\sqrt{33}}{2}$	−	−	+
	$\frac{-1-\sqrt{33}}{2}$	$\frac{-1+\sqrt{33}}{2}$	
$f'(x)$	+	−	+

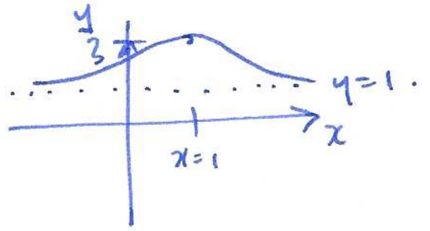
local max $\qquad$ local min

Q10 $f(x) = x^2-4x-4$ $\quad f'(x) = 2x-4$ $\quad$ critical pt $x = 2$

check: $f(-2) = 0$ abs max

$f(2) = -8$ abs min.

Q11



Q12 $f(x) = \dfrac{e^x}{9-x^2}$ $\quad$ a) vertical asymptotes $x = \pm 3$

horizontal asymptotes: $\lim\limits_{x\to\infty} \dfrac{e^x}{9-x^2} = \lim\limits_{x\to\infty} \dfrac{e^x}{-2x} = \lim\limits_{x\to\infty} \dfrac{e^x}{-2} = \infty$

$\lim\limits_{x\to-\infty} \dfrac{e^x}{9-x^2} = \lim\limits_{x\to\infty} \dfrac{e^{-x}}{9-x^2} = \cancel{\lim\limits_{x\to\infty} \dfrac{-e^{-x}}{-2x}} = 0$ $\quad y=0$ left vertical asymptote

b) $f'(x) = \dfrac{(9-x^2)e^x - e^x \cdot (-2x)}{(9-x^2)^2} = \dfrac{e^x(9+2x-x^2)}{(9-x^2)^2}$ $\quad f'(x)=0 \Leftrightarrow x^2-2x-9=0$

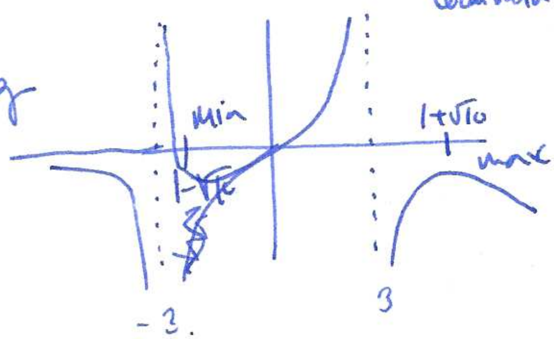
$x = \dfrac{2 \pm \sqrt{4+36}}{2} = 1 \pm \sqrt{10}$.

c) $e^x > 0$, $(9-x^2)^2 > 0$ $\quad f'(x)$: $-$ $1-\sqrt{10}$ $+$ $1+\sqrt{10}$ $-$

local min $\quad$ local max.

d) too long

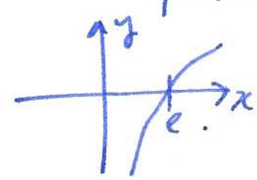
e)



Q13 $g(x) = x\ln x - 2x = x(\ln x - 2)$

a) $g'(x) = 1.\ln x + x.\frac{1}{x} - 2 = \ln x - 1 = 0 \Rightarrow x = e$

$g'(x) = \ln x - 1$



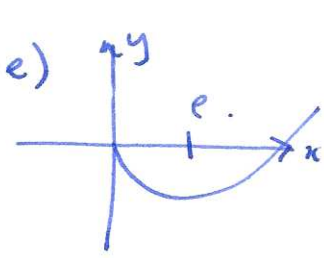
b) $g'(x) < 0$ for $x < e$ $\quad g$ decreasing on $(0,e)$ $\Rightarrow x = e$ local min.

$g'(x) > 0$ for $x > e$ $\quad g$ increasing on (e,∞)

c) $g''(x) = \frac{1}{x}$ ← no points of inflection

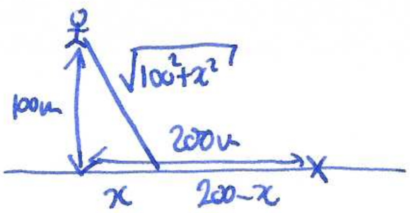
d) $g''(x) > 0$ concave up.

e)



Q14 $f'(x) = \dfrac{1}{e^{x^2}+1} > 0 \Rightarrow f$ increasing, so max on $[1,3]$ at $x=3$.

Q15



$T = \sqrt{100^2+x^2} + \frac{200-x}{2}$ solve $\frac{dT}{dx} = 0$

$\frac{dT}{dx} = \frac{1}{2}(100^2+x^2)^{-1/2} \cdot 2x - \frac{1}{2}$ $\qquad \frac{2x}{\sqrt{100^2+x^2}} = 1$

$2x = \sqrt{100^2+x^2}$ $\quad 4x^2 = 100^2 + x^2$ $\quad 3x^2 = 100^2$ $\quad x = \frac{100}{\sqrt{3}}.$ $\quad x \approx 57.7$

Q16

circumference $2\pi R - \theta R = R(2\pi - \theta)$ $\quad r = \text{radius} = \frac{R(2\pi-\theta)}{2\pi} = R\left(1 - \frac{\theta}{2\pi}\right)$

$h^2 + r^2 = R^2$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi R^2\left(1-\frac{\theta}{2\pi}\right)^2 \cdot \sqrt{R^2 - R^2\left(1-\frac{\theta}{2\pi}\right)^2} = \frac{1}{3}\pi R^3\left(1-\frac{\theta}{2\pi}\right)^2\left(1-\left(1-\frac{\theta}{2\pi}\right)^2\right)^{1/2}$

~~set $x = 1 - \frac{\theta}{2\pi}$ $\quad \frac{dx}{d\theta} = -\frac{1}{2\pi}$ $\quad \frac{dV}{d\theta} = \frac{dV}{dx}\cdot\frac{dx}{d\theta} = \left(\frac{1}{3}\pi R^3 (1-x)^2 (1-(1-x))^{1/2}\right)' \cdot -\frac{1}{2\pi}$~~

$= \frac{1}{3}\pi r^2 \sqrt{R^2 - r^2}$ $\quad \frac{dV}{dr} = \frac{1}{3}\pi 2r\sqrt{R^2-r^2} + \frac{1}{3}\pi R^2 r^2 \frac{1}{2}(R^2-r^2)^{-1/2}\cdot(-2r)$

solve $\frac{dV}{dr} = 0$: $\quad 2r\sqrt{R^2-r^2} = \frac{r^3}{\sqrt{R^2-r^2}}$ $\quad 2(R^2-r^2) = r^2$ $\quad 2R^2 = 3r^2$ $\quad r = \sqrt{\frac{2}{3}}R$

$\sqrt{\frac{2}{3}}R = R\left(1-\frac{\theta}{2\pi}\right) \Rightarrow \frac{\theta}{2\pi} = 1 - \frac{\sqrt{2}}{\sqrt{3}}$ $\quad \theta = 2\pi\left(1 - \sqrt{2/3}\right).$

Q17 a) $\lim_{x\to-\infty} \frac{2-3x}{\sqrt{2x^2-3}} = \lim_{x\to\infty} \frac{2+3x}{\sqrt{2x^2-3}} = \lim_{x\to\infty} \frac{3+2/x}{\sqrt{2-3/x^2}} = \frac{3}{\sqrt{2}}.$

b) $\lim_{x\to 0} \frac{\sin^{-1}(2x)}{\cos^{-1}(3x)} = \frac{0}{1} = 0$

c) $\lim_{x\to 0} \sin x \ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\operatorname{cosec}(x)} \overset{l'H}{=} \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec}(x)\cot(x)} = \lim_{x\to 0} \frac{-\sin x \tan x}{x}$

$\overset{l'H}{=} \lim_{x\to 0} \frac{-\cos x \tan x - \sin x \sec^2 x}{1} = 0.$

d) $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{e^x - 1} = \lim_{x\to 0} \frac{e^x - 1 - \sin x}{\sin x (e^x - 1)} \overset{l'H}{=} \lim_{x\to 0} \frac{e^x - \cos x}{\cos x (e^x - 1) + \sin x e^x}$

$\overset{l'H}{=} \lim_{x\to 0} \frac{e^x + \sin x}{-\sin x(e^x - 1) + \cos x (e^x) + \cos x e^x + \sin x e^x} = \frac{1}{2e} = \frac{1}{2e}.$

e) $\lim_{x\to 0} \frac{3\tan x - \tan 3x}{\sin^2 2x} \overset{l'H}{=} \lim_{x\to 0} \frac{3\sec^2 x - 3\sec^2 3x}{2\sin 2x \cdot \cos 2x \cdot 2}$ $\overset{l'H}{=} \lim_{x\to 0} \frac{6\sec x \cdot \sec x \tan x - 6\sec^2 3x \tan 3x \cdot 3.}{8 \cos 4x} = 0$

$2\sin 4x =$