SMT2 Q7

$$f(x) = \sqrt{x^3} = x^{3/2}$$

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

c center

find ~~g~~ deg 3 Taylor polynomial

- centered at 0 ?
- centered at 1 ?

find an error bound for $\sqrt{2}$.

$$f(x) = x^{3/2}$$

$$f'(x) = \frac{3}{2} x^{1/2}$$

$$f''(x) = \frac{3}{4} x^{-1/2}$$

$$f^{(3)}(x) = -\frac{3}{8} x^{-3/2}.$$

$$f^{(4)}(x) = \frac{9}{16} x^{-5/2}$$

at 0: no, $f^{(2)}(0)$ not defined

$f(1) = 1$

$f'(1) = \frac{3}{2}$

$f''(1) = \frac{3}{4}$

$f^{(3)}(1) = -3/8.$

$$T_3(x) = 1 + \frac{3}{2}(x-1) + \frac{3}{4}\frac{(x-1)^2}{2!}$$

$$+ -\frac{3}{8}\frac{(x-1)^3}{3!}$$

~~$I(x) =$~~ centered at c:

$$T(x) = a_0 + a_1(x-c) + a_2\frac{(x-c)^2}{2!} + \cdots$$

$$a_0 = f(c) \qquad a_1 = \frac{f'(c)}{1!} \qquad a_2 = \frac{f''(c)}{2!}. \qquad a_n = \frac{f^{(n)}(c)}{n!}$$

error bound thm.

$$\left| T_n(x) - f(x) \right| \leq K \cancel{f}\, \frac{|x-c|^{n+1}}{(n+1)!}$$

K is an upper bound for

$f^{(n+1)}(x)$ on $[c, x]$.

$n=3$ center $c=1$.

$x = \cancel{\sqrt{2}}\ 2$. $\quad f(x) = 2^{3/2} = 2\sqrt{2}$.

$$f(x) = x^{3/2} \qquad T_3(2) = 1 + \frac{3}{2} + \frac{3}{4} - \frac{3}{8} = \frac{8+12+8-3}{8}$$

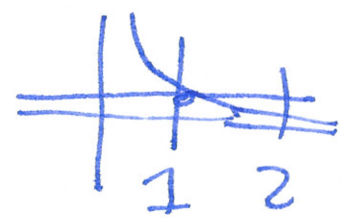
$$= \frac{25}{8}$$

$$\left| 2^{3/2} - \frac{25}{8} \right| \le K \frac{\left|\overset{2-1}{x-c}\right|^{n+1}}{(n+1)!} = \frac{K}{24} = \frac{9/16}{24}.$$

$n=3$.

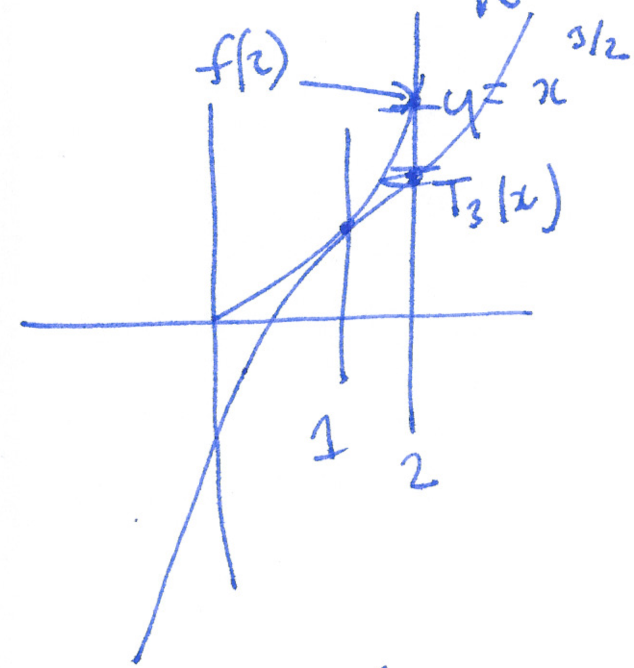
K is an upper bound for $f^{(4)}(x) = \frac{9}{16} x^{-5/2}$

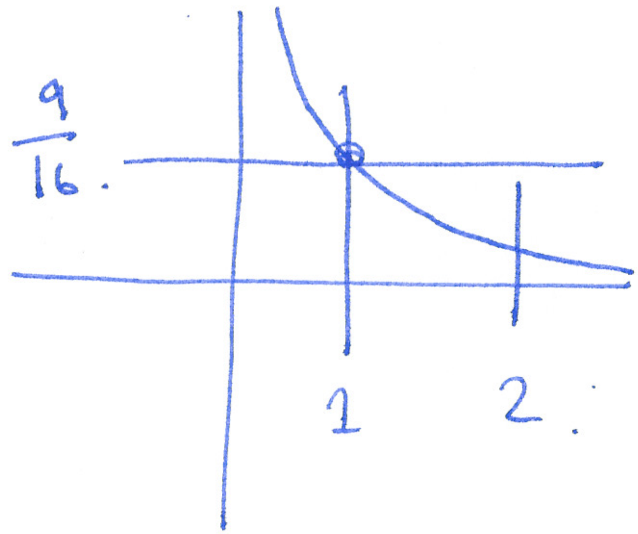
on $[1,2]$. $\qquad K = \frac{9}{16}$.



$f^{(4)}(x) = \frac{9}{16} x^{-5/2}$. upper bound on $[1,2]$.

c x





$f^{(4)}(x) = \frac{9}{16} x^{-5/2}$

$f^{(n)}(x) = 10\sin(x) + x^4 - 7x^2$. upper bound on $[0,10]$.

$|a+b| \le |a| + |b|$.

$|f^{(n)}(x)| \le 10\underbrace{\sin(x)}_{|\cdot| \le 1} + x^4 + 7x^2 \le 10 + 10^4 + 7\times 10^2$.

$f(x) = x^{3/2}$. find $2^{3/2}$.

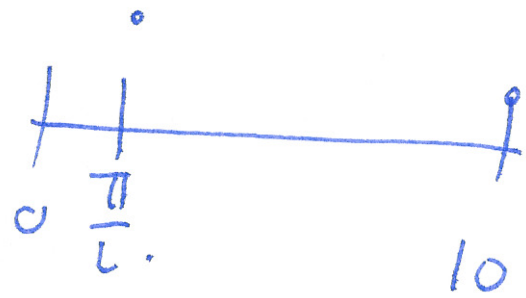
$x = 2$ $f(2) = 2^{3/2}$.

find $\sqrt{2}$ $x = \sqrt[3]{2}$

$\uparrow$

$f(x) = \sqrt{x} \leftarrow$ find Taylor poly at $x = 1$.

$f(x) = 10\sin(x) + \cdots$



$|f(x)| \leq 10$.

$x = 2$. $f(x) = \sqrt{2}$.

want approx for $\sqrt{2}$

$f(x) = x^{1/2}$ $\qquad f(1) = 1$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\qquad f'(1) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$. $\qquad f''(1) = -\frac{1}{4}$

$T_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!}$

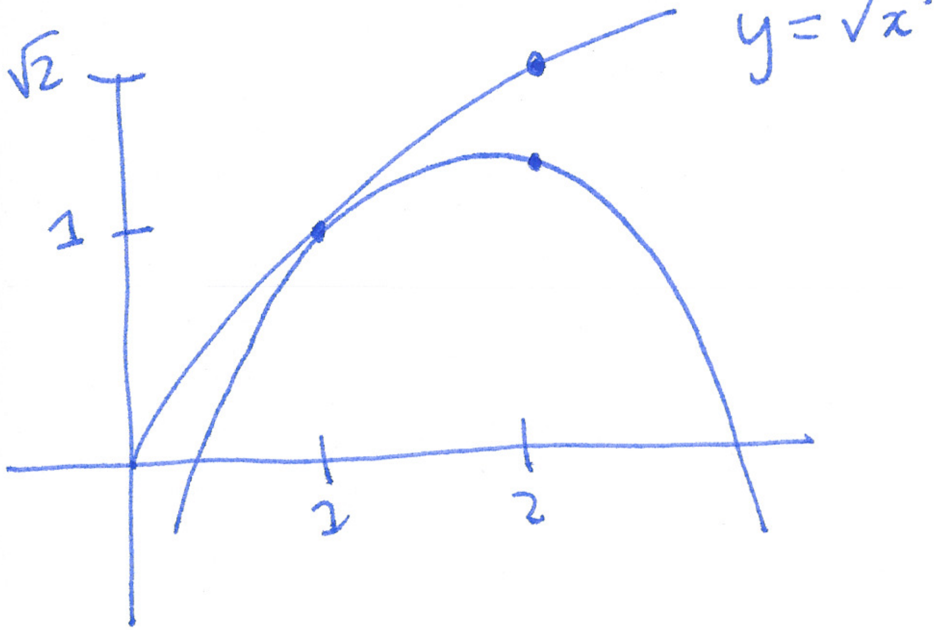
$T_2(2) =$

$1 + \frac{1}{2}(1) - \frac{1}{4}\frac{1}{2}$.

$1 + \frac{1}{2} - \frac{1}{8}$.

$\frac{8+4-1}{8}$. $=$

$\frac{11}{8}$.



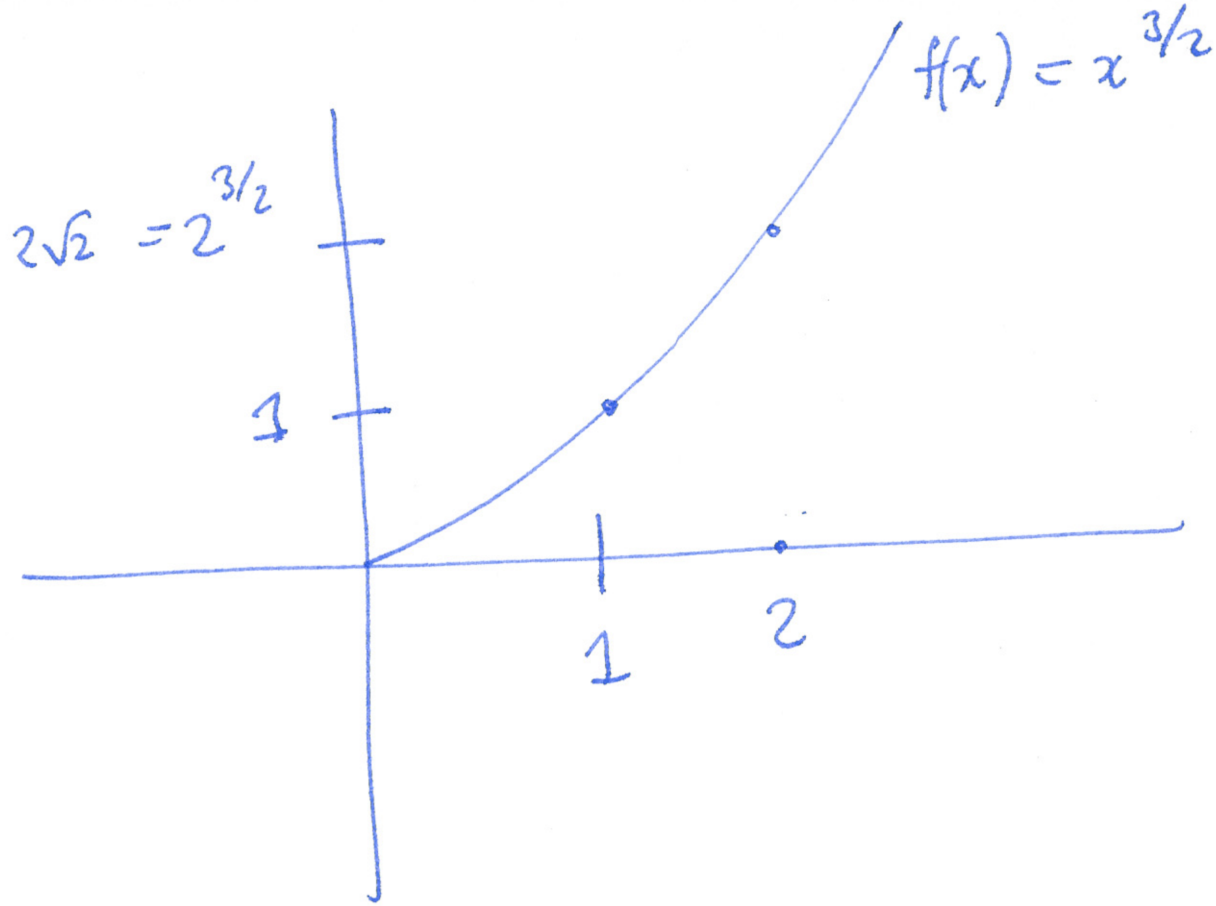
$$f(x) = \sqrt{x}$$

$$T_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!}$$

$$T_2(2) = \frac{11}{8} \approx \sqrt{2}.$$

$f(x) = x^{3/2}$.

$2^{3/2}$



Q10

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+8n+3} = \frac{1}{(2n+1)(2n+3)}.$$

$$\frac{A}{2n+1} + \frac{B}{2n+3} = \frac{A(2n+3)+B(2n+1)}{(2n+1)(2n+3)}.$$

$$1 = A(2n+3) + B(2n+1).$$

$n = -\frac{1}{2}$: $\quad 1 = A2 \qquad A = 1/2.$

$n = -\frac{3}{2}$: $\quad 1 = B(-2) \qquad B = -1/2.$

$$1 = A(2n+3) + B(2n+1).$$

$n = -\frac{1}{2}$: $\quad 1 = A.2 + B.0 \qquad A = \frac{1}{2}.$

$n = -\frac{3}{2}$: $\quad 1 = A.0 + B.(-2) \qquad B = -1/2.$

$$1 = \underbrace{n(2A+2B)}_{0.} + \underbrace{3A + B}_{1.}$$

$$\left.\begin{aligned} 2A + 2B &= 0 \\ 3A + B &= 1 \end{aligned}\right\} \qquad \begin{aligned} A &= 1/2 \\ B &= -1/2. \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{4n^2+8n+3} = \sum_{n=1}^{\infty} \frac{1/2}{2n+1} - \frac{1/2}{2n+3}.$$

$$\boxed{S_N = \frac{1/2}{3} - \frac{1/2}{2N+3}}$$

$$S_1 = \frac{1/2}{3} - \frac{1/2}{5}$$

$$S_2 = \frac{1/2}{3} - \cancel{\frac{1/2}{5}} + \cancel{\frac{1/2}{5}} - \frac{1/2}{7}.$$

$$S_3 = \frac{1/2}{3} - \cancel{\frac{1/2}{5}} + \cancel{\frac{1/2}{5}} - \cancel{\frac{1/2}{7}} + \cancel{\frac{1/2}{7}} - \frac{1/2}{9}.$$

$$S_N = \frac{1/2}{3} - \cancel{\frac{1/2}{5}} + \cancel{\frac{1/2}{5}} \cdots\cdots - \cancel{\frac{1/2}{2N+1}} + \cancel{\frac{1/2}{2N+1}} - \frac{1/2}{2N+3}$$

$$\sum_{n=1}^{\infty} \frac{1/2}{2n+1} - \frac{1/2}{2n+3}.$$

$$S_N = \frac{1/2}{3} - \cancel{\frac{1/2}{5}} + \cancel{\frac{1/2}{5}} - \cancel{\frac{1/2}{7}} + \cdots + \cancel{\frac{1/2}{2N+1}} - \frac{1/2}{2N+3}$$

$$S_N = \frac{1/2}{3} - \frac{1/2}{2N+3}.$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{6} - \frac{1}{4N+6} = \frac{1}{6}.$$

Q9 $\displaystyle\sum_{n=2}^{\infty} e^{-n} = e^{-2} + e^{-3} + e^{-4} + \cdots\cdots$

$$= \frac{1}{e^2} + \frac{1}{e^3} + \frac{1}{e^4} + \cdots\cdots$$

$a + ar + ar^2 + ar^3 + \cdots\cdots$ $\quad \dfrac{a}{1-r}$ converges $|r| < 1$.

diverges if $|r| \geq 1$.

$a = e^{-2}$

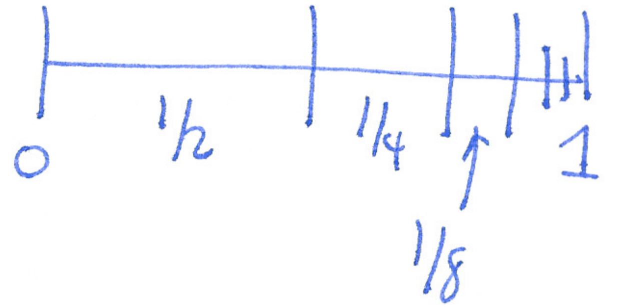
$r = \dfrac{1}{e} < 1$

$\cancel{\displaystyle\sum_{n=2}^{\infty} \frac{1/e^2}{1-1/e} =}$

$$\sum_{n=2}^{\infty} e^{-n} = \frac{1/e^2}{1 - 1/e} = \frac{1}{e^2 - e}$$

$$a + ar + ar^2 + ar^3 + \cdots$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$



$r = 1/2$

$r = -\frac{1}{2}$. $$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots = \frac{1/2}{1-(-1/2)} = \frac{1/2}{1+1/2}$$

$$\frac{1}{2} + \frac{1}{8} + \cdots \quad - \left(\frac{1}{4} - \frac{1}{16} \cdots\right). \qquad = \frac{1/2}{3/2} = \frac{1}{3}.$$

$r = 2$. $$2 + 2^2 + 2^3 + \cdots$$

$$2 + 4 + 8 + 16 + \cdots$$

$r = -2$ $$2 - 4 + 8 - 16 + \cdots$$

$$a + ar + ar^2 + ar^3 + \cdots = a/1-r$$

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots\cdots = \frac{1/2}{1-(-1/2)}.$$

$a = \frac{1}{2}$. $r = -\frac{1}{2}$.

$$\frac{1/2}{1+1/2} = \frac{1}{3}.$$

$S_0 = a$

$S_1 = a + ar$

$$S_N = a + ar + ar^2 + \cdots + ar^N$$

$$rS_N = ar + ar^2 + ar^3 + \cdots + ar^{N+1}.$$

$$S_N - rS_N = a - ar^{N+1}$$

$$S_N(1-r) = a(1-r^{N+1}).$$

$$S_N = \frac{a(1-r^{N+1})}{1-r}$$

$$\lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r} = \frac{a}{1-r} \quad \text{if} \quad |r| < 1.$$

Q16 $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ use ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| .$$

$$= \lim_{n\to\infty} |x| \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} |x| \frac{1}{(1+1/n)^2} = |x| .$$

converges if $|x| < 1$. $x = 1$

diverges if $|x| > 1$. $x = -1$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$\underline{x=1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

converges (p-series w/ $p > 1$).

$\underline{x=-1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{2^2} - \frac{1}{3^2} + \cdots$

converges by alternating series test

$(-1)^n \frac{1}{n^2}$. check. $\frac{1}{n^2}$ positive ✓

decreasing ✓

converges to 0 as $n \to \infty$ ✓.

Q11 $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converge or diverge?

if $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.

if $a_n \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ does not converge.

$$\cos\left(\frac{1}{n}\right) \to \underset{\neq 0}{1} \quad \text{as} \quad n \to \infty$$

so diverges.

warning about (sloppy) language.

· converges. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

$$\sum_{n=1}^{\infty} 2^n = 2+4+8+\cdots$$ ~~converges to $+\infty$.~~

diverges to $+\infty$.

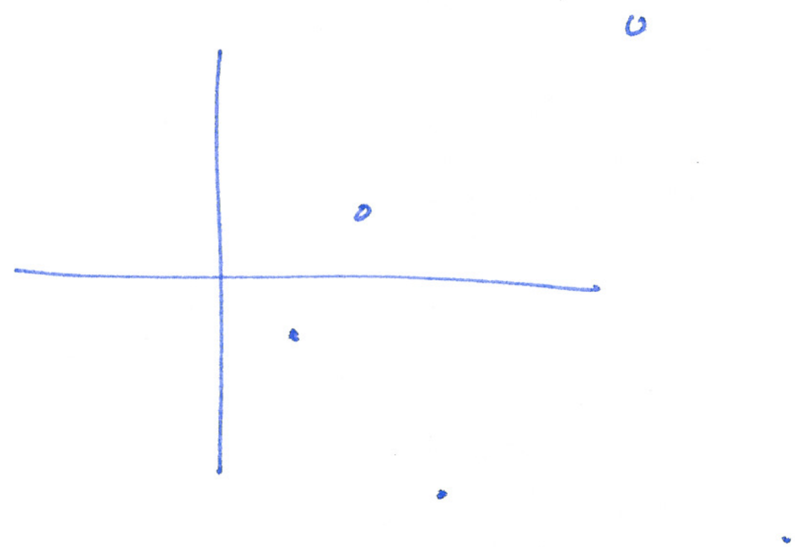
$$\sum_{n=1}^{\infty} (-2)^n = -2+4-8+16-32+\cdots$$ diverges.

$S_1 = -2$

$S_2 = 2$

$S_3 = -6$

$S_4 = 10$



Q12 $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^4}$ converge or diverge?

try limit comparison test $a_n = \dfrac{(\ln n)^2}{n^4}$

$b_n = \dfrac{1}{n^2}$.

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(\ln n)^2}{n^4} \cdot n^2 \right|$$

$$= \lim_{n\to\infty} \frac{(\ln n)^2}{n^2} \overset{L'H}{=} \left(\lim_{n\to\infty} \frac{\ln(n)}{n} \right)^2.$$

L'Hôpital: $\displaystyle\lim_{n\to\infty} \frac{1/n}{1} = 0.$ $\qquad 0.$

If limit is 0 $\qquad \sum b_n$ converge

$\Rightarrow \sum a_n$ converge.

$f(x)$ is continuous at L and $a_n \to L$

then $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} (a_n)\right)$.

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n^2} \overset{l'H}{=} \lim_{n\to\infty} \frac{2\ln(n)\cdot 1/n}{2n} = \lim_{n\to\infty} \frac{\ln(n)}{n^2}.$$

$$\overset{l'H}{=} \lim_{n\to\infty} \frac{1/n}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0.$$

$$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^4}$$

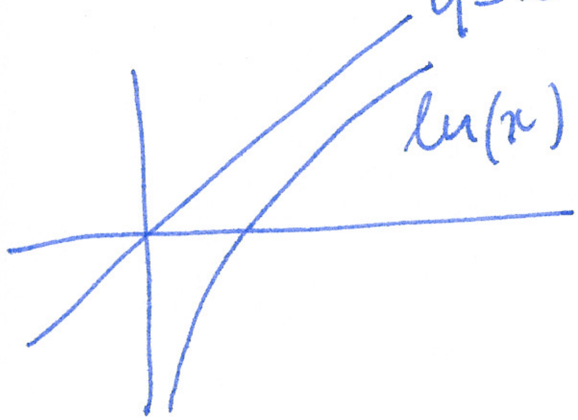
$b_n = \frac{1}{n^3}$ work

$b_n = \frac{1}{n^4}$ doesn't work.

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = \lim_{n\to\infty} \left| \frac{(\ln n)^2}{n^4} \cdot n^4 \right| = \lim_{n\to\infty} |(\ln n)^2| \to \infty.$$

no information

(needs +ve series)
comparison test



$$a_n = \frac{(\ln n)^2}{n^4}.$$

know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

$$x > \ln(x)$$

$$x^2 > (\ln(x))^2.$$

$$(\ln(n))^2 < n^2$$

$$\frac{(\ln(n))^2}{n^4} < \frac{n^2}{n^4}$$

$$\frac{1}{n^2}.$$

comparison. $\sum a_n$ $\sum b_n$. +ve series.

$a_n < b_n$ and $\sum b_n$ converge $\Rightarrow$ $\sum a_n$ converge

$a_n < b_n$ and $\sum a_n$ diverge $\Rightarrow$ $\sum b_n$ diverge.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ $\}$ ratio test $\Rightarrow$ converges for all x.

ratio test: (any series, not nec positive).

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \begin{aligned} &< 1 \quad \text{converges} \\ &> 1 \quad \text{diverges} \\ &= 1 \quad \text{no information.} \end{aligned}$$

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limit comparison test (positive series).

$$\lim_{n\to\infty} \left| \frac{a_n}{b_n} \right| = L$$

if $0 < L < \infty$

then $\sum a_n$ iff $\sum b_n$ converges.

$L = 0$ $\quad \sum b_n$ converge $\Rightarrow$ $\sum a_n$ converge.

Q3. $\int \frac{x}{\sqrt{16x^2+1}}\, dx$.

- trig sub.
- comes from a chain rule

$u = 16x^2+1$

$\frac{du}{dx} = 32x$.

$\int \frac{x}{\sqrt{u}} \cdot \frac{dx}{du}\, du$

$\int \frac{\cancel{x}}{\sqrt{u}} \frac{1}{32\cancel{x}}\, du = \frac{1}{32}\int u^{-1/2}\, du = \frac{1}{32} 2 u^{1/2}$

$= \frac{1}{16} u^{1/2} + c = \frac{1}{16}\sqrt{16x^2+1} + c$.

$\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ diverges $\cos\left(\frac{1}{n}\right) \to 1$.

$\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$. $\frac{1}{n} \to 0$ $\sin(0) = 0$.

↑ diverge, $\boxed{\sin\left(\frac{1}{n}\right) \sim \frac{1}{n}}$ for large n.

- show $\sin\left(\frac{1}{n}\right)$ is positive for $\cancel{n \gg 0}$. $n \gg 0$

- limit comparison w/ $\frac{1}{n}$. $\sum \frac{1}{n}$ diverges.

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0} \frac{\sin(x)}{x} \overset{\text{l'H}}{=} 1.$$

$\sin(x) \approx x.$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$$