§11.3 Polar coordinates

WW 10.8 Q2

$$f(x) = \cos(17x).$$

$$f'(x) = -17\sin(17x).$$

$$\sum_{n=0}^{\infty} a_n x^n.$$

$$f''(x) = -17^2\cos(17x)$$

centered at 0

$$\Leftrightarrow \quad a_n = \frac{f^{(n)}(0)}{n!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(17x) = 1 - \frac{(17x)^2}{2!} + \frac{(17x)^4}{4!} - \frac{(17x)^6}{6!} + \cdots$$

$$\sum_{n=0}^{\infty} a_n x^n$$ power series.

f(x), center c $\rightsquigarrow$ $\displaystyle\sum_{n=0}^{\infty} a_n \cancel{x^n}\,(x-c)^n$. where

$$a_n = \frac{f^{(n)}(c)}{n!}$$

$$\cos(17x) = 1 - \frac{(17x)^2}{2!} + \frac{(17x)^4}{4!} - \frac{(17x)^6}{6!} + \cdots$$

$$\sum_{n=0}^{\infty}$$

n	0	1	2	3	4
	1	0	$\frac{-17^2}{2!}$	0	$\frac{17^4}{4!}$

n	0	1	2	3	4
	1	$\frac{-17^2}{2!}$	$\frac{17^4}{4!}$	…	

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$$\cos(17x) = \underset{0}{1} - \underset{1}{\frac{(17x)^2}{2!}} + \underset{2}{\frac{(17x)^4}{4!}} \cdots \underset{n}{(-1)^n \frac{(17x)^{2n}}{(2n)!}}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(17x)^{2n}}{(2n)!}$$

find interval of convergence.

ratio test
$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \cancel{(-1)^{n+1}} \frac{(17x)^{2n+2}}{(2n+2)!} \frac{(2n)!}{(17x)^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| (17x)^2 \frac{(2n)!}{(2n+2)!} \right| = \lim_{n\to\infty} \left| \frac{17^2 x^2}{(2n+1)(2n+2)} \right| = 0 < 1$$

converges for all x, $R = \infty$, $(-\infty, \infty)$.

Warning $\sum_{n=0}^{\infty} a_n$ $\sum_{n=0}^{\infty} a_n x^n$

ratio test : $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ suppose this limit exists.

then $\sum_{n=0}^{\infty} a_n$ converges if $L < 1$.

diverges if $L > 1$

no information $L = 1$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| .$$

$$\frac{(2n)!}{(2n+2)!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot \ldots \cancel{(2n)}}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot \ldots \cancel{(2n)}(2n+1)(2n+2)}.$$

$$= \frac{1}{(2n+1)(2n+2)}.$$

warning $\quad 2(n!) \neq (2n)! \qquad 2n!$

$n! \quad 1\cdot 2\cdot 3\cdot \ldots\ldots n.$

$3! \quad 1\cdot 2\cdot 3.$

$6! \quad 1\cdot 2\cdot 3\cdot \ldots 6.$

remark $\quad 2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots \quad (2n)$.

$2\times 1 \quad 2\times 2 \quad 2\times 3 \quad 2\times 4 \quad \ldots \quad 2\times n$

$$2^n(1 \cdot 2 \cdot 3 \cdot \ldots \; n) = 2^n \cdot n!$$

$f(x) = x^2$.

$1 \cdot 3 \cdot 5 \cdot \ldots \quad (2n+1)$

$(n)! = 1 \cdot 2 \cdot 3 \cdot \ldots \quad n$.

$((2n)!+2)! \quad 1 \cdot 2 \cdot 3 \cdot \ldots \quad ((2n)!+2)$.

$n=2$. $\quad (4!+2)! \quad 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 26$.

$(24+2)!$

$\boxed{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1).}$ $\qquad (2n+2)!$

$$\frac{(2n+2)!}{2^n (n!)} = \frac{1 \cdot \cancel{2} \cdot 3 \cancel{4} \ldots (2n+1)\,\cancel{(2n+2)}}{\cancel{2} \cdot \cancel{4} \cdot \cancel{6} \ldots \cancel{(2n+2)}} = 1.3.5 \ldots (2n+1)$$

$1.3.5 \ldots . (n).$ here I mean n to be odd.

↑ n-th odd number. n.

$(2n+1)$.

<u>observation</u> $\quad (n!)(n+1) = (n+1)!$

10.8 Q6.

$$f(x) = \frac{1}{1-x^2} \quad \text{centered at } c=3.$$

$$\sum_{n=0}^{\infty} a_n (x-3)^n$$

normally:

$$\left.\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\ \frac{1}{1-x^2} &= 1 + x^2 + x^4 + x^6 + \cdots \end{aligned}\right\} \text{not centered at 3 !}$$

$f(x) = \dfrac{1}{1-x}$ has power series $\sum_{n=0}^{\infty} (-2)^{-n-1} (x-3)^n$

$f(x) = \dfrac{1}{1-x^2}$

$f(x) = x.$

$f(x) = (x-3) + 3.$

$f(x) = \dfrac{1}{1 - ((x-3)+3)^2} = \dfrac{1}{1 - (x-3)^2 - 6(x-3) \cancel{+} - 9}.$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}.$$

$$f'(x) = +(1-x)^{-2}.$$

$$f''(x) = 2(1-x)^{-3}.$$

$$f^{(3)}(x) = 2.3\,(1-x)^{-4}.$$

$$f^{(n)}(x) = n!\,(1-x)^{-(n+1)}$$

$$\sum_{n=0}^{\infty} a_n (x-3)^n \qquad a_n = \frac{f^{(n)}(3)}{n!}$$

$$a_n = \frac{n!\,(1-3)^{-n-1}}{n!} = (-2)^{-n-1}.$$

$$f(x) = \frac{1}{1-x^2} = (1-x^2)^{-1}$$

$$f'(x) = +(1-x^2)^{-2} \cdot 2x$$

$$f''(x) = 2(1-x^2)^{-3} \cdot 2x + (1-x^2)^{-2} \cdot 2 \quad \cdots$$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x} = \frac{1/2}{1-x} + \frac{1/2}{1+x}$$

$$\boxed{\frac{1}{2}\sum_{n=0}^{\infty} (-2)^{-n-1}(x-3)^n} = \sum_{n=0}^{\infty} \frac{1}{2}(-2)^{-n-1}(x-3)^n$$

$$f(x) = \frac{1}{1+x} = (1+x)^{-1}.$$

$$f'(x) = -(1+x)^{-2}$$

$$f''(x) = 2(1+x)^{-3}$$

$$f^{(3)}(x) = 2\,3(1+x)^{-4}.$$

$$f^{(n)}(x) = n!\,(-1)^n(1+x)^{-n-1}$$

$$a_n = \frac{f^{(n)}(3)}{n!}$$

$$\sum_{n=0}^{\infty} (-1)^n (4)^{-n-1} (x-3)^n$$

$$f(x) = \frac{1}{1-x^2} = \frac{1/2}{1-x} + \frac{1/2}{1+x}. = \boxed{\frac{\cancel{n!}(-1)^n(4)^{-n-1}}{\cancel{n!}}(x-3)^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(-2)^{-n-1}(x-3)^n + \sum_{n=0}^{\infty}(-1)^n(4)^{-n-1}(x-3)^n.$$

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$$\sum_{n=0}^{\infty} \frac{1}{2}\left((-2)^{-n-1} + (-1)^n (4)^{-n-1}\right)(x-3)^n .$$

$$\boxed{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

use ratio test

$$\sum_{n=0}^{\infty} \frac{1}{2}(-2)^{-n-1}(x-3)^n + \sum_{n=0}^{\infty}\frac{1}{2}(-1)^n (4)^{-n-1}(x-3)^n .$$

use ratio test. use ratio test.

$$\lim_{n\to\infty}\left|\frac{\cancel{\frac{1}{2}}(-2)^{-n-2}(x-3)^{n+1}}{\cancel{\frac{1}{2}}(-2)^{-n-1}(x-3)^n}\right| \qquad \lim_{n\to\infty}\left|\frac{\frac{1}{2}(-1)^{n+1}(4)^{-n-2}(x-3)^{n+1}}{\frac{1}{2}(-1)^n(4)^{-n-1}(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\frac{1}{2}|x-3| . \qquad \lim_{n\to\infty}\frac{1}{4}|x-3| < 1 .$$

$$= \frac{1}{2}|x-3| < 1 .$$

$|x-3| < \cancel{\tfrac{1}{2}}\, 2$ $\quad$ $|x-3| < \cancel{\tfrac{1}{4}}.\ 4$

$R = 2.$

~~$R = 1/4.$~~ $\quad$ ~~$\left(3-\tfrac{1}{4},\ 3+\tfrac{1}{4}\right).$~~

$(3-2,\ 3+2).$

$(1,\ 5).$

check endpoints manually.

$[\qquad)$

$(\qquad].$

$[\qquad].$

10.8 Q3:

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$f(x) = \ln(1-x^3)$. centered at 0.

$\ln(1-x)$.

$f(x) = \ln(1-x)$.

$f'(x) = \frac{-1}{1-x}$

$1 + x + x^2 + x^3 + \cdots \quad \frac{1}{1-x}$

$-1 - x - x^2 - x^3 + \cdots = \frac{-1}{1-x}$. $\Leftarrow$ integrate

$-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots = \ln(1-x)$.

$$\ln(1-x^3) = -x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \frac{x^{12}}{4} - \cdots - \frac{x^{3n}}{n}.$$

$$\sum_{n=1}^{\infty} -\frac{x^{3n}}{n}.$$

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{3(n+1)}}{n+1} \cdot \frac{n}{x^{3n}} \right|.$

$$= \lim_{n\to\infty} |x^3| \underbrace{\frac{n}{n+1}}_{\to 1}. = |x^3| = |x|^3 < 1.$$

$$|x| < 1.$$

$R = 1.$ $(-1,1).$ check end points.

$$-x^3 - \frac{x^6}{2} - \frac{x^9}{3} - \cdots$$

$x = 1$: $\quad -1 - \frac{1}{2} - \frac{1}{3} \cdots$ diverges.

$x = -1$: $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ alternating series test

$\Rightarrow$ converges

$[-1, 1)$.

typo in SMT2 : Q10. $\frac{1}{4u^2+8u+3}$. <u>Q7</u> $2^{3/2}$. (18)

<u>Q7</u>. $f(x) = x^{3/2}$ centered at $x=1$ $f(1)=1$

$f'(x) = \frac{3}{2}x^{1/2}$ $\qquad f'(1) = \frac{3}{2}$.

$f''(x) = \frac{3}{4}x^{-1/2}$ $\qquad f''(1) = 3/4$.

$f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$. $\qquad f^{(3)}(1) = -3/8$.

$$1 + \frac{3}{2}x + \frac{3}{4}\frac{x^2}{2!} - \frac{3}{8}\cdot\frac{x^3}{3!} = T_3(x).$$

$f(2) = 2^{3/2} \simeq T_3(2)$.

error bound: $\left| T_3(\underset{\substack{\uparrow \\ 2}}{x}) - f(x) \right| \le \dfrac{K\,|x-c|^{n+1}}{(n+1)!}$

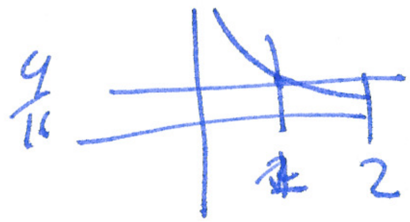
$c = 1$.

$n = 3$.

K = upper bound for $\left| f^{(n+1)}(x) \right|$ on $[1, 2]$.

$[x, c]$.

$f^{(4)}(x) = \dfrac{9}{16}\, x^{-5/2}$.

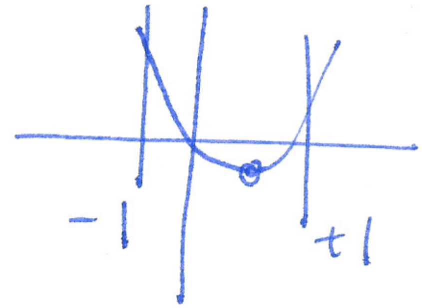


can choose $\dfrac{9}{16}$.

error bound: $\dfrac{\frac{9}{16}\,|2-1|^4}{4!} = \dfrac{9}{16 \times 24}$.

find upper bound.

$$|x^2 - 2x + 1| \text{ on } [-1, 1].$$



$$|a+b| \le |a| + |b|.$$

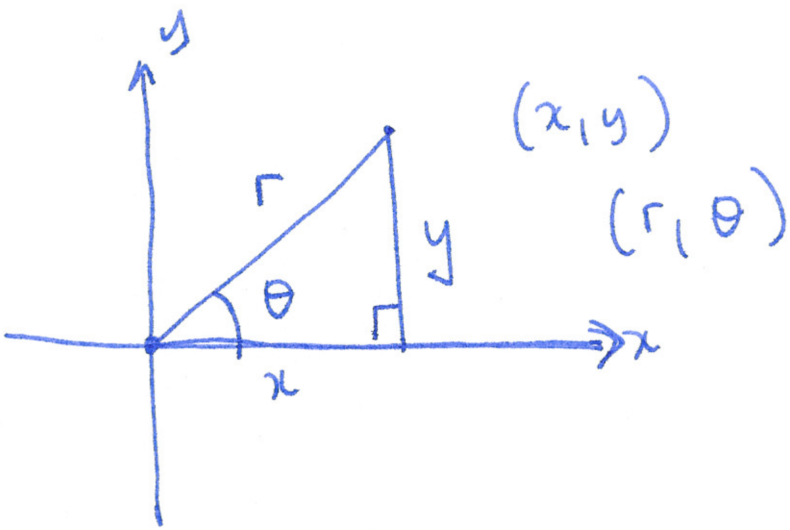
$$|x^2 - 2x + 1| \le |x^2| + |2x| + 1 \text{ on } [-1, 1].$$

$$1 + 2 + 1 = 4.$$

$$|x^2 - 2x + 1| \text{ on } [0, 10].$$

$$\le |x|^2 + 2|x| + 1 \le 100 + 20 + 1 = 121.$$

§ 11.3 Polar coordinates

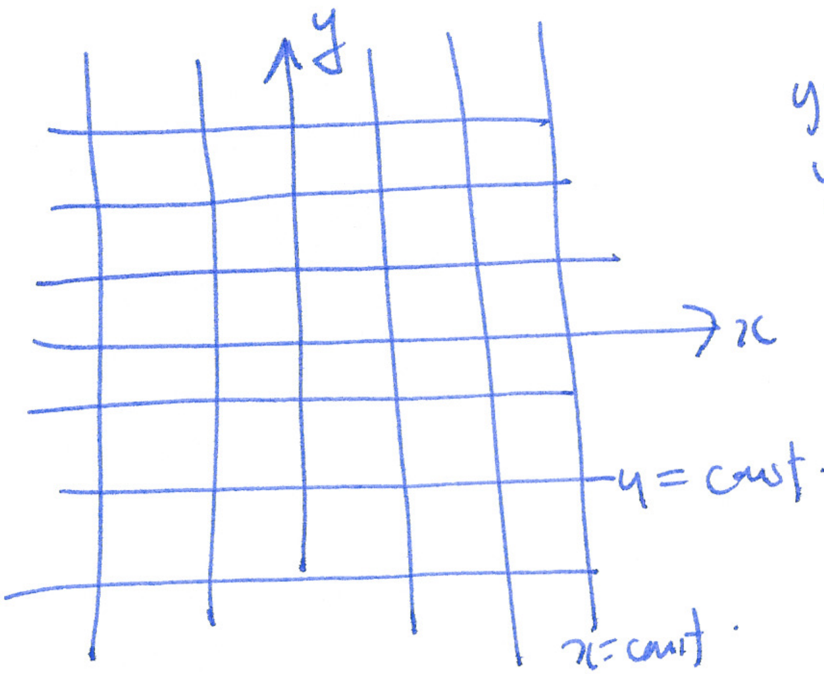


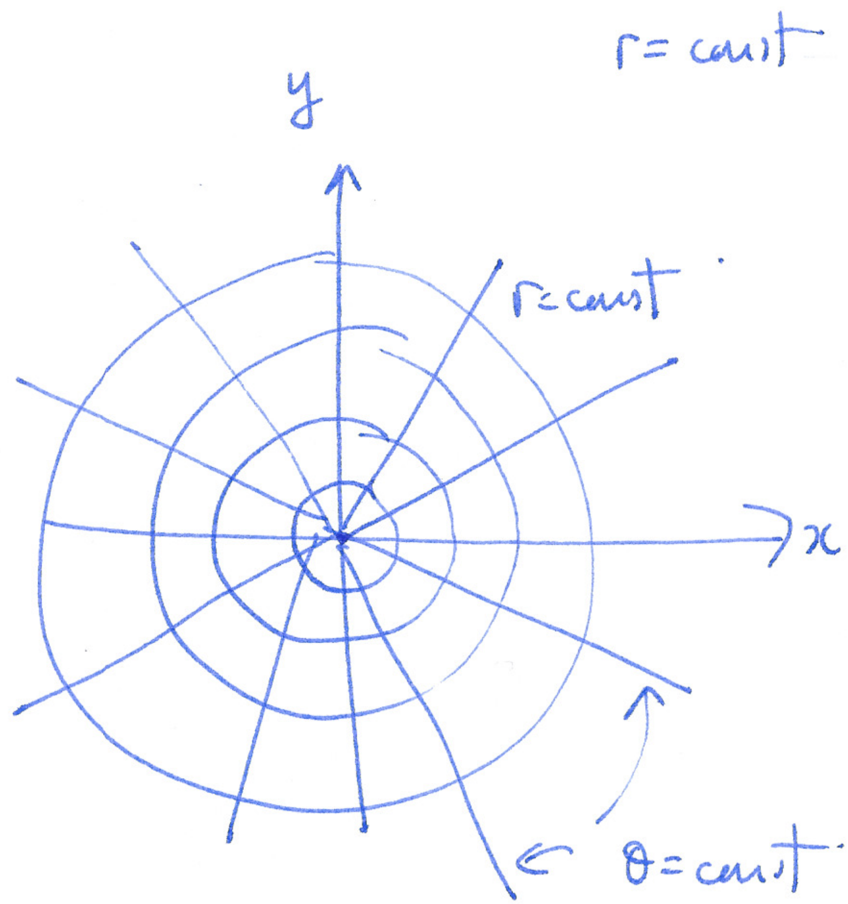
$$x = r\cos\theta \qquad r^2 = x^2 + y^2.$$

$$y = r\sin\theta \qquad \tan\theta = y/x.$$

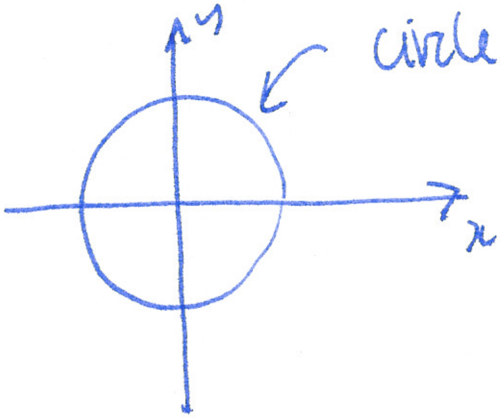
$x = \text{const}.$
$x = 1$

$y = \text{const}$
$y = 1$





Equations in polar coordinates



circle of radius R.

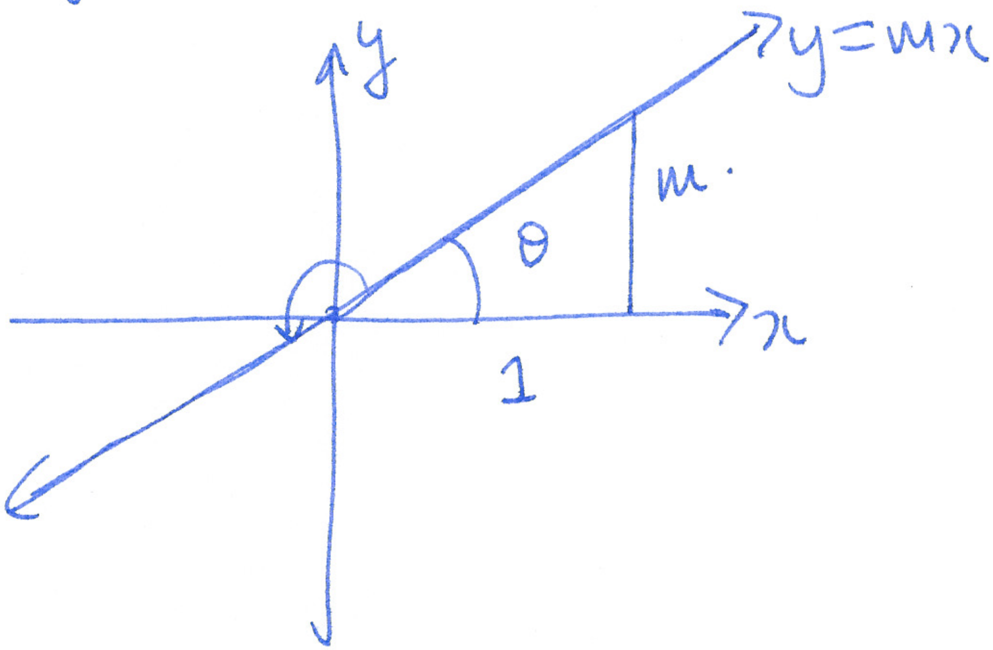
$x^2+y^2=R^2$ cartesian

$r=R.$ polar

straight lines through origin.

$\tan(\theta)=m.$

$\theta=\tan^{-1}(m)$



$$\left.\begin{aligned}\theta&=\tan^{-1}(m)\\ \theta&=\tan^{-1}(m)+\pi.\end{aligned}\right\}.$$

convention : $(-r, \theta) \longleftrightarrow (r, \theta+\pi)$. ($*$)

(r, θ) many people choose $0 \le r < \infty$.

$r \neq 0$ points have unique coords $\theta \in [0, 2\pi)$.

(r, θ) ($*$) convention $-\infty < r < \infty$ $r \in \mathbb{R}$.

$(-r, \theta) \leftrightarrow (r, \theta + \pi)$. $\theta \in [0, 2\pi)$.

(r, θ). another convention $r \in \mathbb{R}$

$(r, \theta) \leftrightarrow (r, \theta + 2\pi)$. $\theta \in \mathbb{R}$

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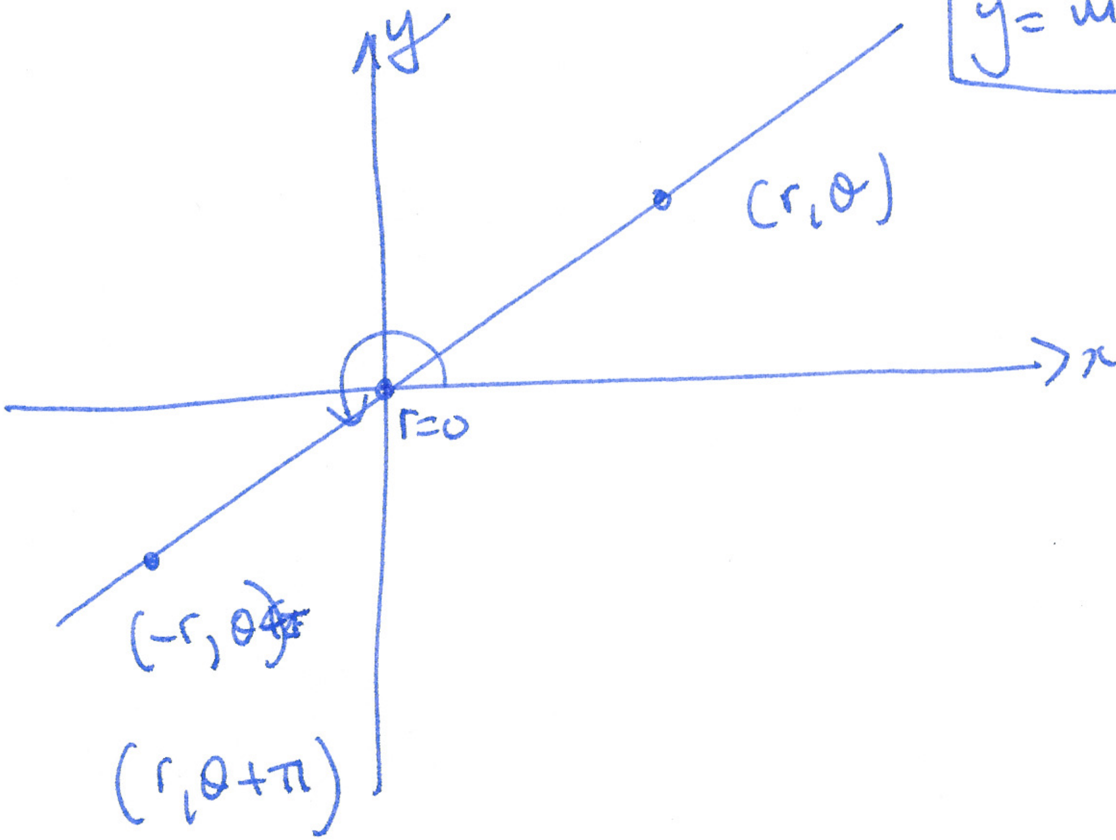
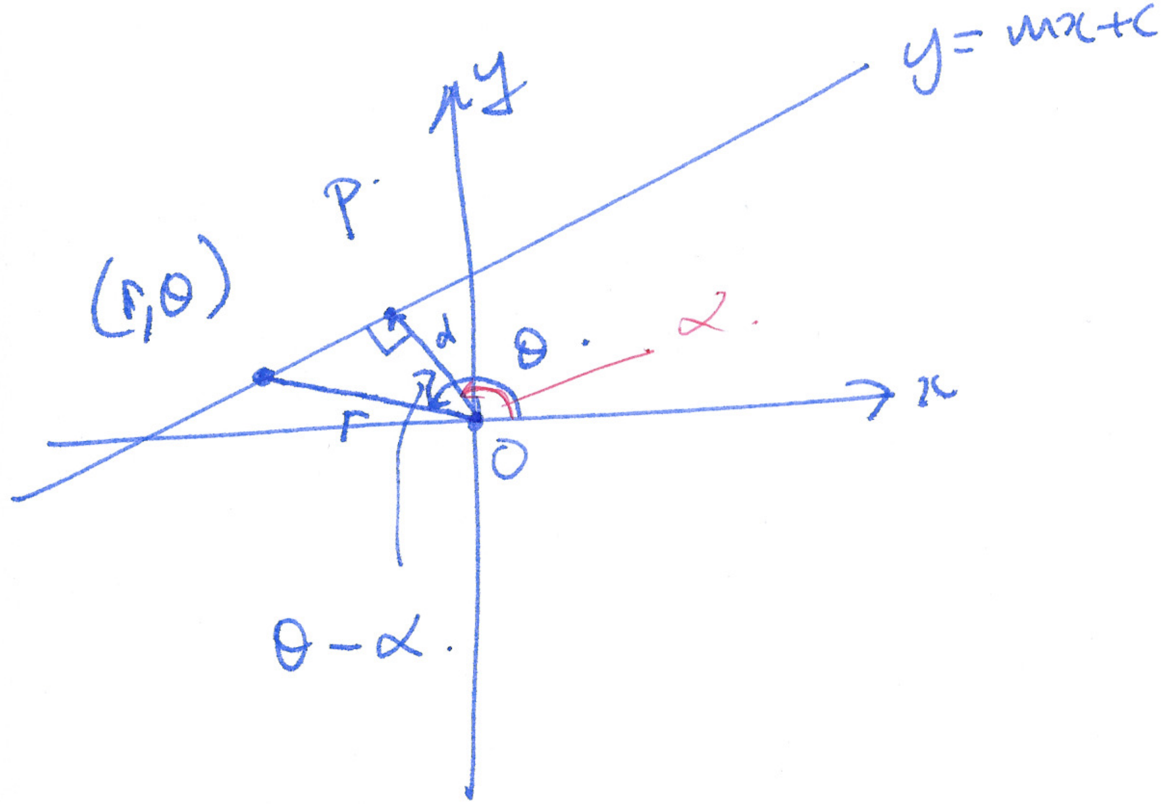
convention $(-r,\theta) \leftrightarrow (r,\theta+\pi)$. $\longrightarrow$ means

$\boxed{y = mx} \leftrightarrow \boxed{r = \tan^{-1}(m)}$



general line $\boxed{y = mx + c}$



Let P be the closest point on line to $(0,0) = O$.

d = distance from P to O.

α = angle of P from O.

$$\frac{d}{r} = \cos(\theta - \alpha)$$

$$\boxed{r = \frac{d}{\cos(\theta - \alpha)} = d \sec(\theta - \alpha)}$$