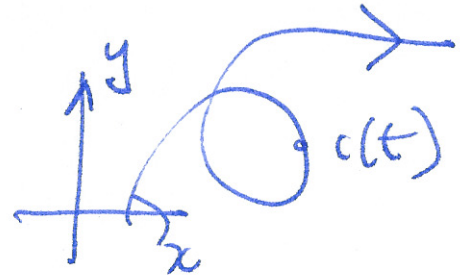
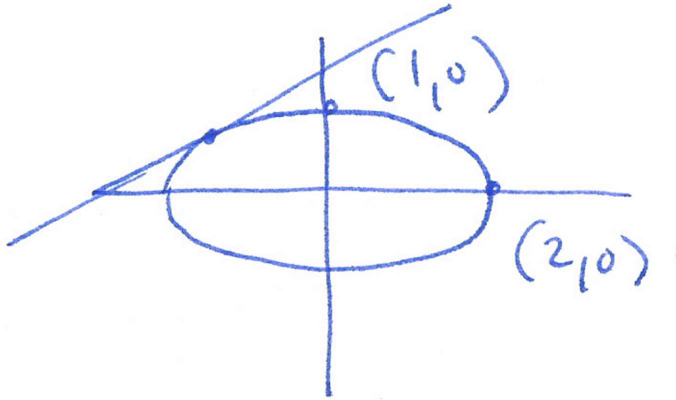
<u>parameterizations</u> $c(t) = (x(t), y(t))$.



<u>Example</u>.

ellipse.



$$c(\theta) = (2\cos\theta, \sin\theta)$$
$$c(t) = (2\cos t, \sin t).$$

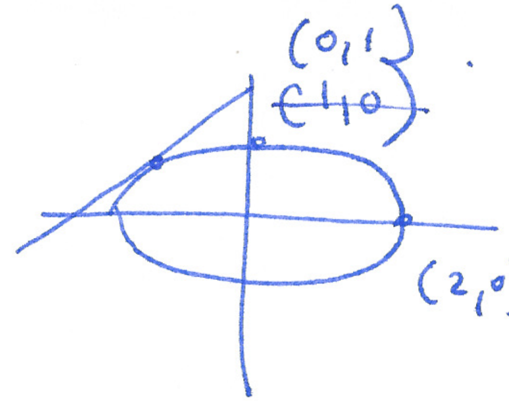
Q: how do we find the tangent line?

A: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt}$ $c(t)$

why: $c(t) = (x(t), y(t))$ chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}.$$

$c(t) = (2\cos(t), \sin(t))$.



$x(t) = 2\cos(t)$ $\qquad$ $y(t) = \sin(t)$

$\dfrac{dx}{dt} = -2\sin(t)$ $\qquad$ $\dfrac{dy}{dt} = \cos(t)$

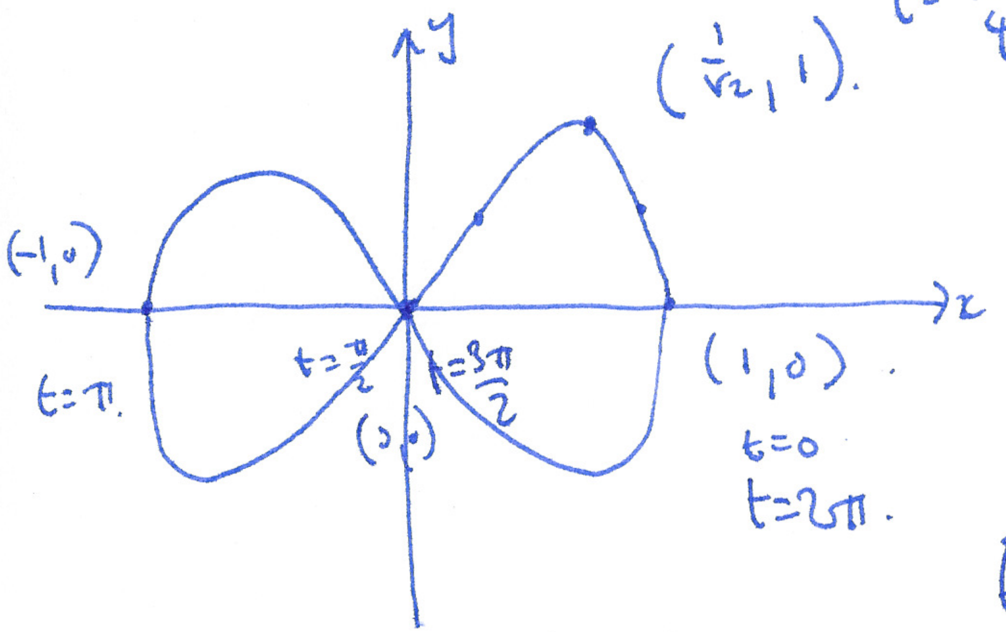
$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\cos(t)}{-2\sin(t)} = -\frac{1}{2}\cot(t) = \frac{\frac{1}{2}x}{-2y}.$$

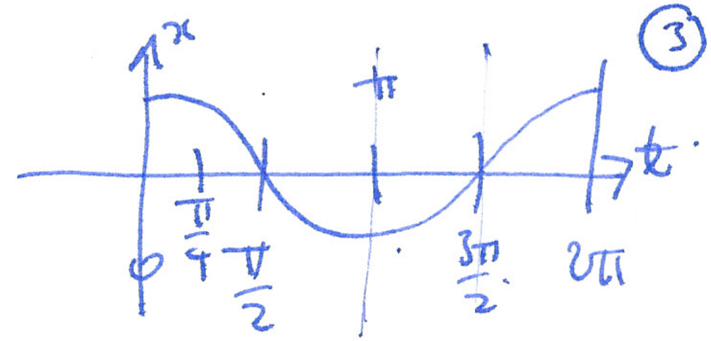
<u>Example</u> $\quad c(t) = (\underset{x(t)}{\cos t}, \underset{y(t)}{\sin 2t})$

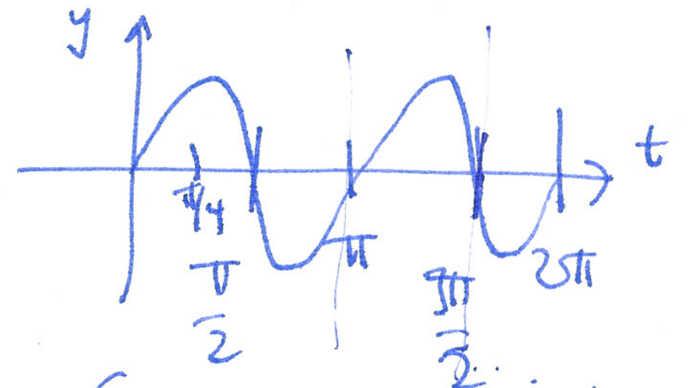
$\dfrac{dx}{dt} = -\sin t$ $\qquad$ $\dfrac{dy}{dt} = 2\cos 2t$ $\qquad$ $\dfrac{dy}{dx} = \dfrac{2\cos 2t}{-\sin t}$

$c(t) = (\cos t, \sin 2t)$

$\cos(t)$





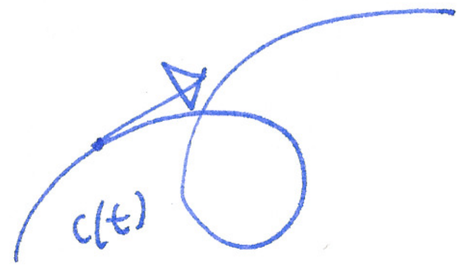


Exercise $(\cos nt, \sin mt)$.

$t=0$. $(\cos(0), \sin(0))$. $(1,0)$.

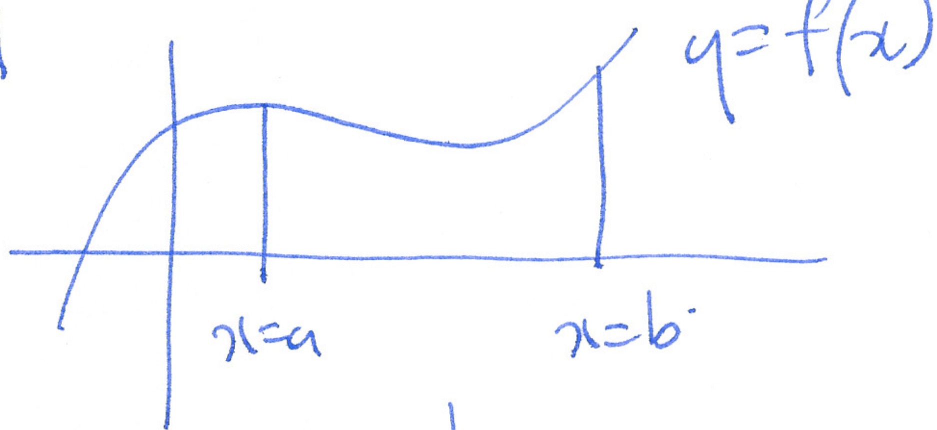
$t=\frac{\pi}{2}$ $(\cos(\frac{\pi}{2}), \sin(2\times\frac{\pi}{2}))$ $(0,0)$.

§11.2 Arc length and speed



$$c(t) = (x(t), y(t))$$
$$c'(t) = (x'(t), y'(t))$$
↑ velocity vector.

recall



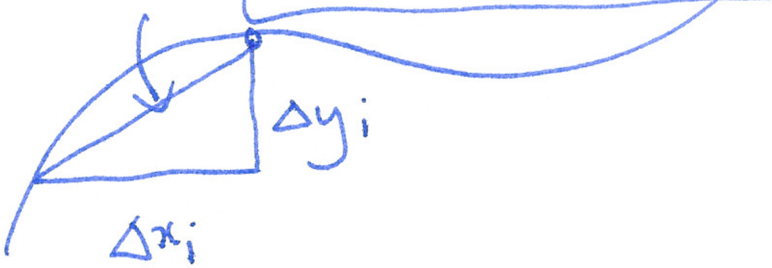
arc length $= \displaystyle\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

↑ parameterized version.

speed $\|c'(t)\|$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\boxed{c(t) = (x(t), y(t))}$$



length $\displaystyle\sum_{i=1}^{n} \Delta L_i$.

$$\text{arc length} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}\,.$$

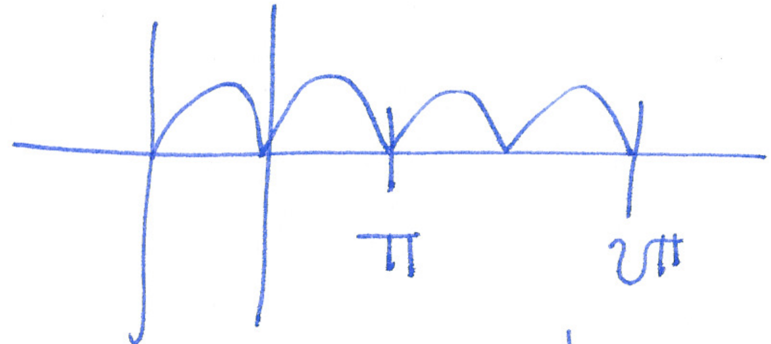
$$= \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\ \Delta t_i\,.$$

$$\boxed{\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,.}$$

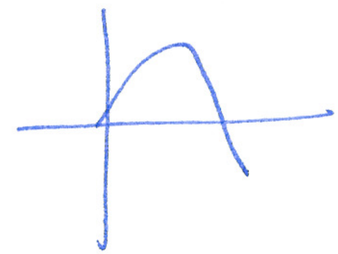
8

$$\int_0^{2\pi} 2\left|\sin\left(\frac{t}{2}\right)\right| dt = \cancel{\left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 0}.$$

$$4\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{t}{2}\right) dt$$



$$8\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{\pi/2}.$$

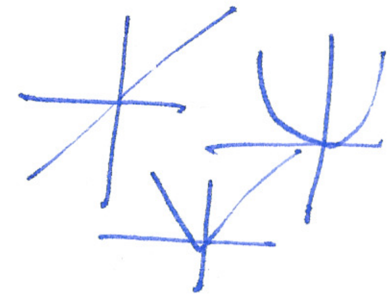


$$8\Big($$

$$\int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt.$$

$$\sqrt{x^2} = |x|$$

$$\int_0^{2\pi} \sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1.}}\, dt.$$



recall

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$
$$= 1 - 2\sin^2\theta.$$

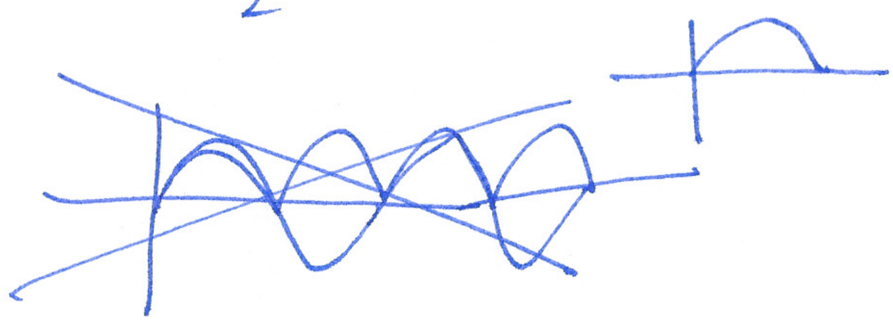
$$\int_0^{2\pi} \sqrt{\underset{2(1-\cos t).}{2 - 2\cos t}}\, dt.$$

$$2\sin^2\theta = 1 - \cos 2\theta.$$

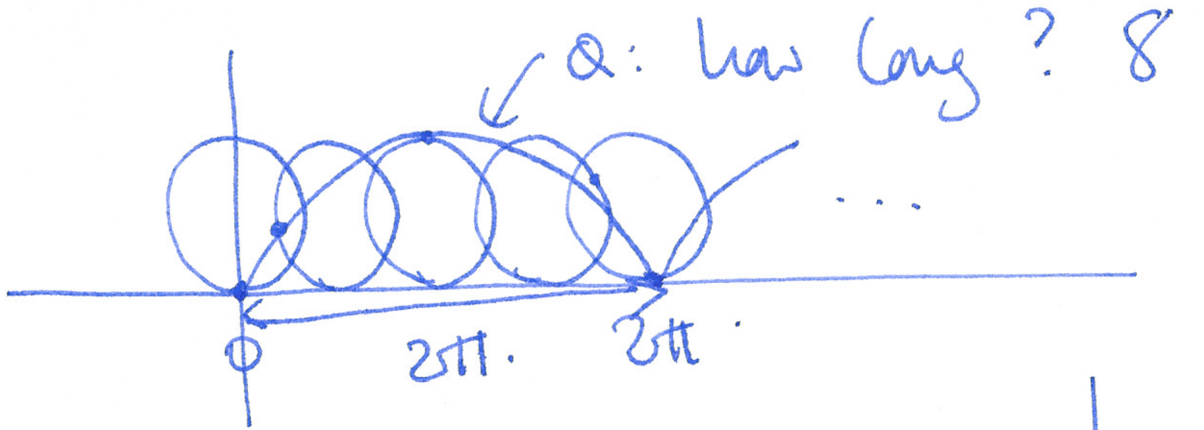
$$2\sin^2\frac{t}{2} = 1 - \cos t$$

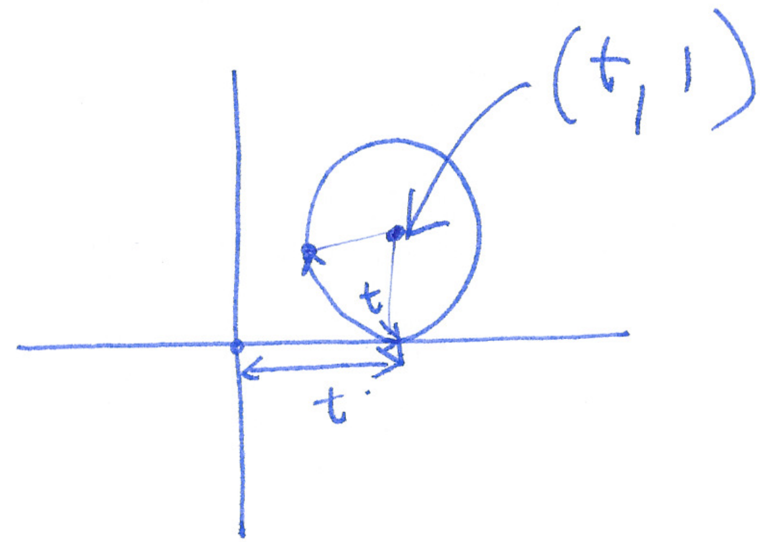
$$\int_0^{2\pi} \sqrt{2\left(2\sin^2\frac{t}{2}\right)}\, dt.$$

$$\int_0^{2\pi} 2\left|\sin\frac{t}{2}\right| dt.$$



Example





$$x(t) = t - \sin t$$

$$y(t) = 1 - \cos t$$

$$\frac{dx}{dt} = 1 - \cos t$$

$$\frac{dy}{dt} = \sin t$$

arc length

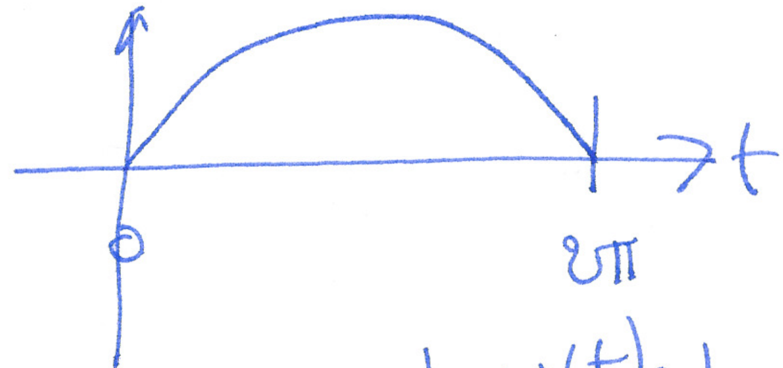
$$\int_0^{2\pi} \sqrt{\cancel{1+}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

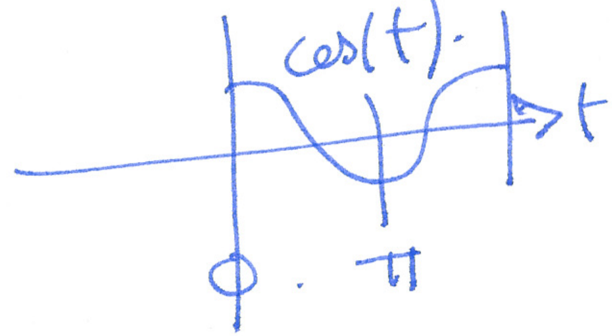
$$\left[-4\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}.$$

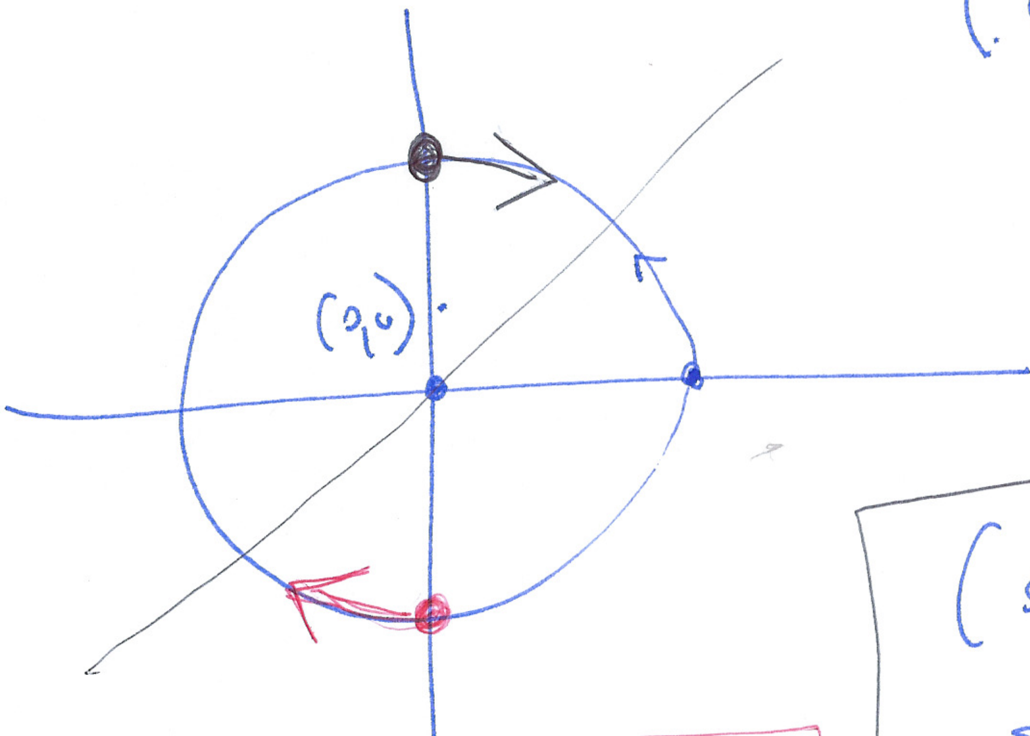
$$-4\left(\cos(\pi)-\cos(0)\right).$$

$$-4(-1-1) = 8.$$

$\sin\left(\frac{t}{2}\right).$







$(\cos t, \sin t)$.

start $t = 0$

$(1, 0)$.

anti-clockwise.

$(\sin t, \cos t)$.

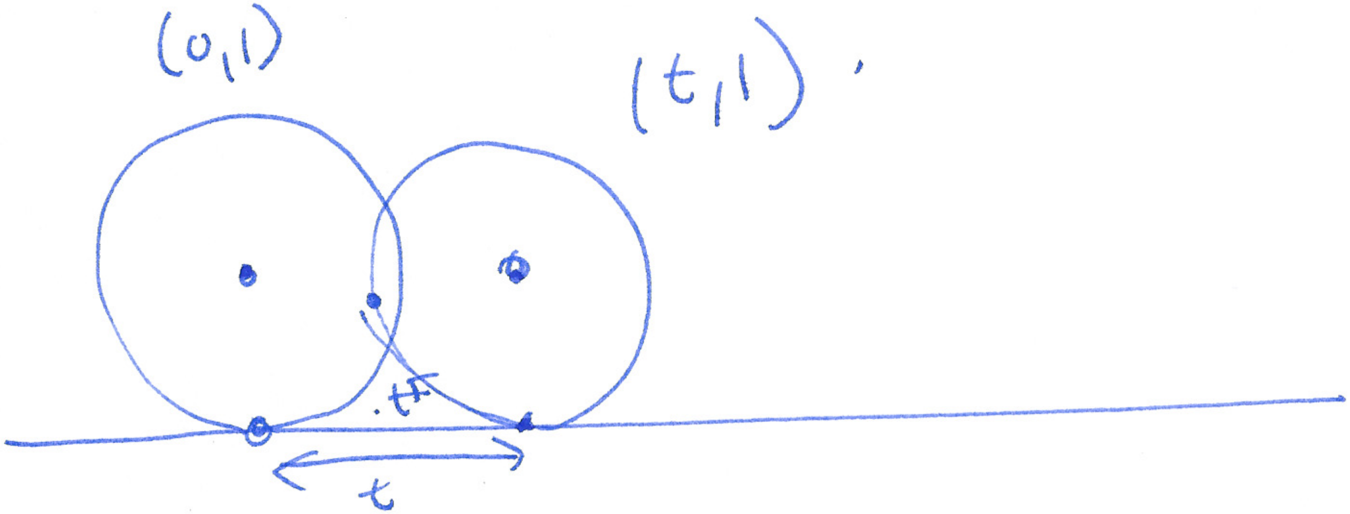
start $t=0$ $(0, 1)$

clockwise.

start $(0, -1)$

$(-\sin t, -\cos t)$

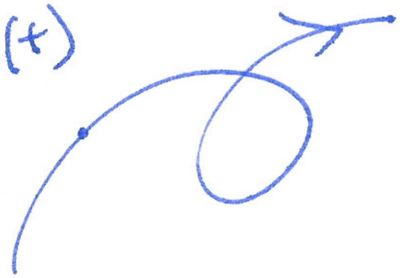
clockwise



$$x(t) = t - \sin t$$

$$y(t) = 1 - \cos t$$

speed. $c(t)$

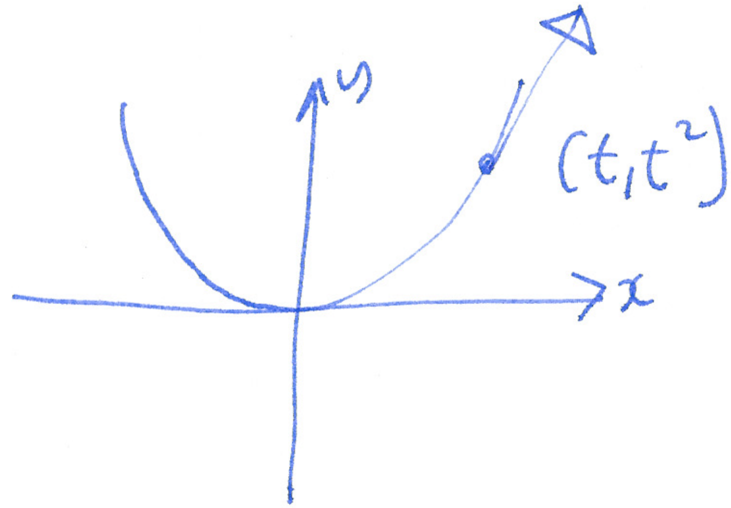


$$c(t) = (x(t), y(t)).$$

$$c'(t) = (x'(t), y'(t)).$$

speed $\|c'(t)\| = \sqrt{(x')^2 + (y')^2}.$

Example.

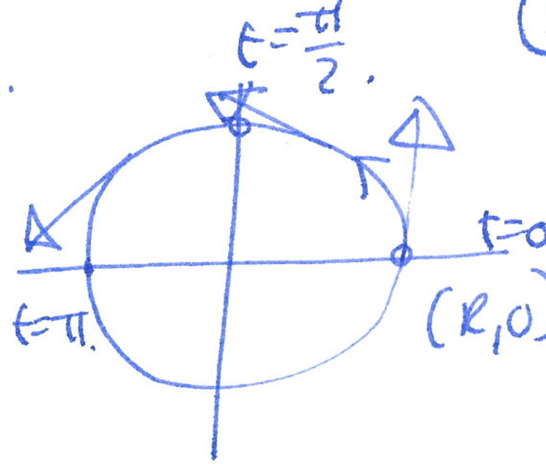


(1) $c(t) = (t, t^2)$

$$c'(t) = (1, 2t).$$

$$\|c'(t)\| = \sqrt{1 + 4t^2}.$$

<u>Example</u>. $c(t) = \left(R\cos(\underset{\substack{\uparrow \\ \text{angular velocity } \omega}}{\omega} t), R\sin(\omega t) \right).$



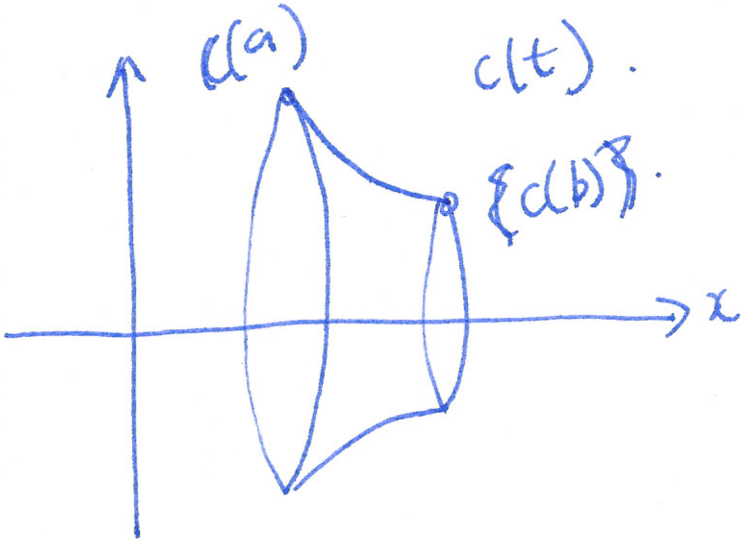
$$c'(t) = \left(-R\omega \sin(\omega t), R\omega \cos(\omega t) \right).$$

$$\|c'(t)\| = \sqrt{R^2\omega^2 \sin^2\omega t + R^2\omega^2\cos^2\omega t}.$$

$$= \sqrt{R^2\omega^2 \underbrace{(\sin^2\omega t + \cos^2\omega t)}_{1.}}$$

$$= \sqrt{R^2\omega^2} = R|\omega|.$$

§11.3. Surface area for parameterized curves of revolution

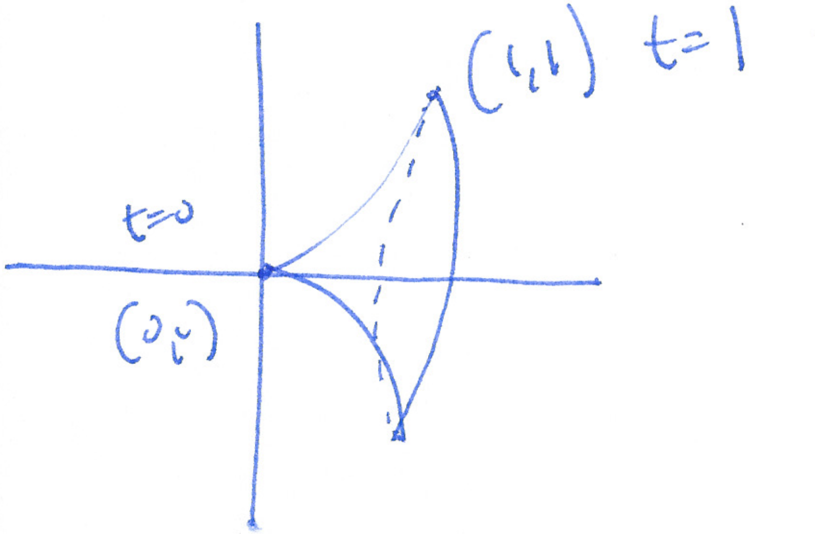


$c(t)$. $a \le t \le b$.

surface area

$$\boxed{2\pi \int_a^b y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Example. $c(t) = (t, t^3)$. $0 \le t \le 1$



$c'(t) = (1, 3t^2)$.

$\frac{dx}{dt}$ $\frac{dy}{dt}$.

15

$$2\pi \int_0^1 y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$2\pi \int_0^1 t^3 \sqrt{1 + (3t^2)^2} \, dt$$

$$2\pi \int_{t=0}^{t=1} t^3 \sqrt{1 + 9t^4} \, dt$$

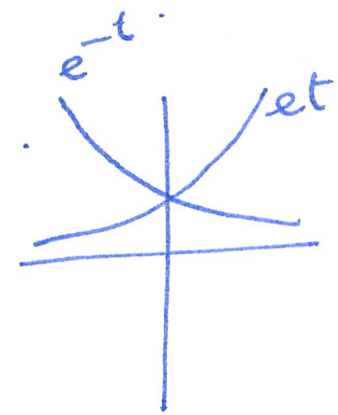
$$u = 1 + 9t^4$$

$$\frac{du}{dt} = 36t^3$$

$$2\pi \int_{u=1}^{u=10} \cancel{t^3} \sqrt{u} \; \boxed{\frac{d\cancel{u}\,t}{d\cancel{t}\,u}} \; d\cancel{t}\,u$$

$$\frac{1}{36\cancel{t^3}}$$

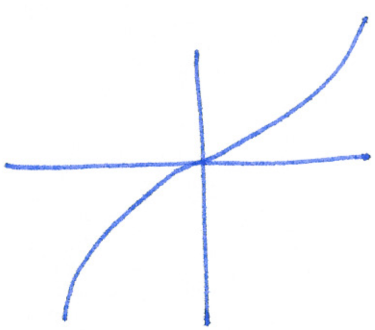
$$2\pi \int_{1}^{10} \frac{1}{36} u^{1/2}\, du = \left[\frac{\pi}{18} \frac{2}{3} u^{3/2} \right]_{1}^{10}.$$

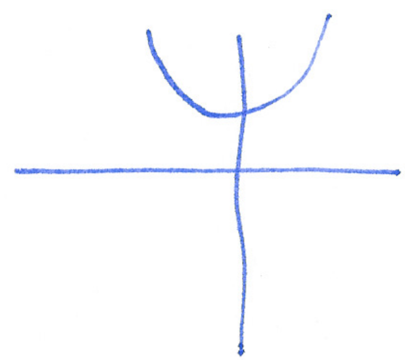


$$= \frac{\pi}{27}\left(10^{3/2} - 1\right) \approx 2.56.$$

<u>Example.</u> $c(t) = \left(t - \tanh(t),\ \operatorname{sech}(t) \right).$

$$\sinh(t) = \frac{e^t - e^{-t}}{2}. \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}.$$





$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} \qquad \operatorname{sech}(t) = \frac{1}{\cosh(t)}$$

$$\coth(t) = \frac{1}{\tanh(t)} \qquad \operatorname{cosech}(t) = \frac{1}{\sinh(t)}.$$

$$\sinh(t) = \frac{e^t - e^{-t}}{2} \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}.$$

$$\sinh^2(t) = \frac{e^{2t} - 2 + e^{-2t}}{4}. \qquad \cosh^2(t) = \frac{e^{2t} + 2 + e^{-2t}}{4}.$$

$$\cosh^2(t) - \sinh^2(t) = 1. \qquad \cos^2\theta + \sin^2\theta = 1.$$

$$\cosh^2(t) - \sinh^2(t) = 1$$

$$\cosh^2(t) = 1 + \sinh^2(t).$$

<u>Remark</u>. $$y = \sinh(t) = \frac{e^t - e^{-t}}{2}$$

$$\frac{dy}{dt} = \cosh(t) = \frac{e^t + e^{-t}}{2}$$

$$\frac{d^2y}{dt} = \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

$y = e^x.$

$y' = e^x$

$\vdots$

1-periodic

$y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y^{(3)} = -\cos x$

$y^{(4)} = \sin x$

4-periodic

$y = \sinh(x)$

$y' = \cosh(x)$

$y'' = \sinh(x).$

2-periodic.

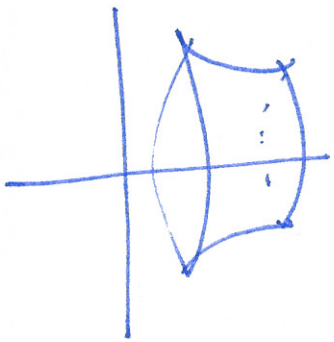
<u>Q</u>: find one that is 3-periodic.

Example. $c(t) = \left(t - \tanh(t),\ \operatorname{sech}(t) \right).$

$$\tanh(t) \uparrow \frac{\sinh(t)}{\cosh(t)} \uparrow \frac{e^t - e^{-t}}{e^t + e^{-t}}$$

$$\operatorname{sech}(t) \uparrow \frac{1}{\cosh(t)} \uparrow \frac{2}{e^t + e^{-t}}.$$

$0 \le t \le 1$



find surface area

$$\int_0^1 2\pi\, y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$\int_0^1 2\pi\, y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \qquad c(t)=\left(t-\tanh t,\ \operatorname{sech}(t)\right)$$

$$\int_0^1 2\pi\,\operatorname{sech}(t)\sqrt{\left(1-\operatorname{sech}^2(t)\right)^2+\left(-\tanh(t)\operatorname{sech}(t)\right)^2}\,dt$$

$$\int_0^1 2\pi\,\operatorname{sech}(t)\sqrt{1-2\operatorname{sech}^2(t)+\operatorname{sech}^4(t)+\underbrace{\tanh^2(t)}_{1-\operatorname{sech}^2(t).}\operatorname{sech}^2(t)}\,dt.$$

$$\cancel{\frac{\cosh^2 t-\sinh^2 t}{\sinh(t)}=\frac{1}{\sinh(t)}} \qquad \cancel{\tan}$$

$$c(t) = \left(t - \tanh(t) \,,\ \operatorname{sech}(t) \right).$$

$$x(t) = t - \frac{\sinh(t)}{\cosh(t)} \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

$$\frac{dx}{dt} = 1 - \frac{\overbrace{\cosh^2(t) - \sinh^2(t)}^{1.}}{\cosh^2(t)} = 1 - \frac{1}{\cosh^2(t)} = 1 - \operatorname{sech}^2(t)$$

$$\frac{dy}{dt} = \frac{1}{\cosh(t)} = \left(\cosh(t)\right)^{-1} = -\left(\cosh(t)\right)^{-2} \cdot \sinh(t).$$

$$= -\frac{\sinh(t)}{\cosh^2(t)} = -\tanh(t)\operatorname{sech}(t).$$

$$\frac{\cosh^2(t) - \sinh^2(t)}{\cosh^2(t)} = \frac{1}{\cosh^2(t)}$$

$$1 - \tanh^2(t) = \operatorname{sech}^2(t).$$

$$\tanh^2(t) = 1 - \operatorname{sech}^2(t)$$

$$\int_0^1 2\pi \operatorname{sech}(t) \sqrt{1 - 2\operatorname{sech}^2(t) + \cancel{\operatorname{sech}^4(t)}}\, dt$$

$$(1 - \cancel{\operatorname{sech}^2(t)})(\operatorname{sech}^2(t)).$$

$$\int_0^1 2\pi \operatorname{sech}(t) \sqrt{1 - \operatorname{sech}^2(t)}\, dt$$

$$\int_0^1 2\pi \operatorname{sech}(t) \sqrt{\tanh^2(t)}\, dt$$

$$\int_0^1 2\pi \operatorname{sech}(t)\tanh(t)\,dt.$$

$$\Big[-2\pi \operatorname{sech}(t)\Big]_0^1 =$$

$$2\pi \operatorname{sech}(0) - 2\pi \operatorname{sech}(1).$$