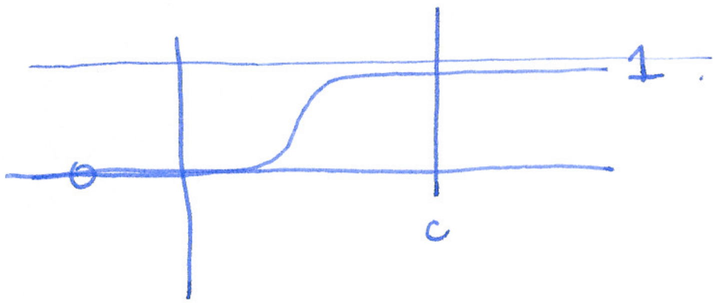
last time: power series: $a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ center c.

defines a function where it converges.

given $f(x) \rightsquigarrow$ Taylor series $a_n = \dfrac{f^{(n)}(c)}{n!}$ n-th derivative.

<u>warning</u> sometimes the power series is zero.

<u>example</u>



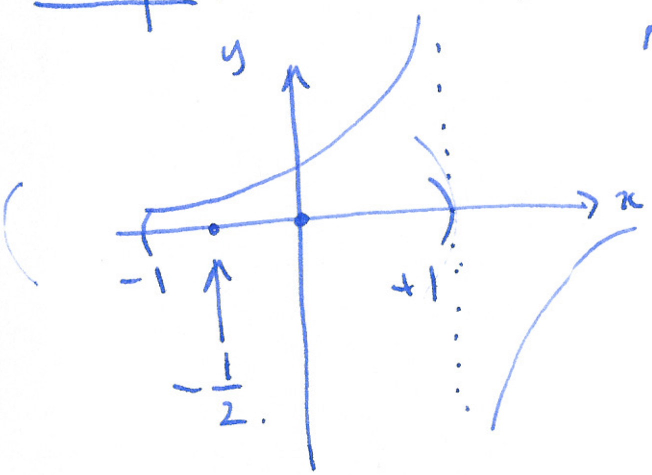
$f(c) = 1$.
$f'(c) = 0$
$f''(c) = 0$
$\vdots$

power series at c is

$1 + 0x + \cdots$

<u>example</u>. $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$.

converges for $|x| < 1$.



could find power series at $c = -\frac{1}{2}$.

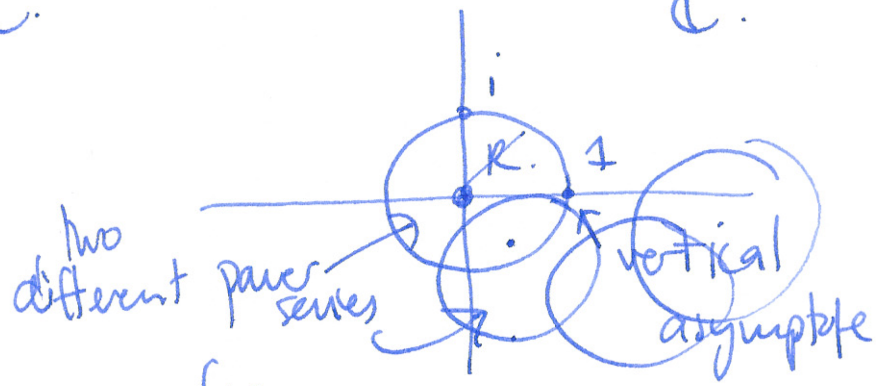
$a_0 + a_1(x+\frac{1}{2}) + a_2(x+\frac{1}{2})^2 + \cdots$

$R = 3/2$.

fact: power series work for $z \in \mathbb{C}$.

$$1+x+x^2+\cdots = \frac{1}{1-x}$$

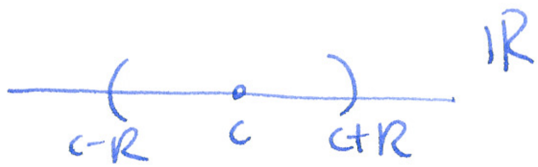
converges for $|z|<1$.



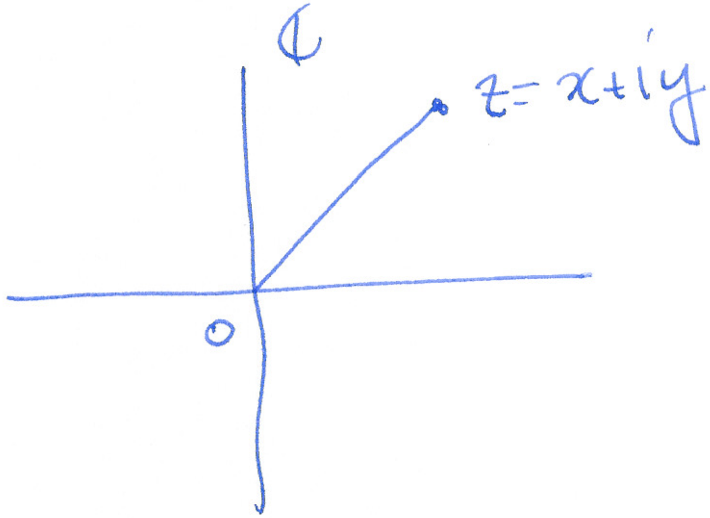
$f(x)$ $\quad f:\mathbb{R}\to\mathbb{R}$.

$f:\mathbb{C}\to\mathbb{C}$. $\quad$ ✓ converges for $|z|<1$.

$z \longmapsto 1+z^1+z^2+z^3+\cdots$



$z = x+iy \qquad |z| = \sqrt{x^2+y^2}$.



$$|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0. \end{cases}$$

- How to make l'Hôpitals rule problems:

$$\sin(2x) = 2x - \frac{8x^3}{3!} + \cdots \qquad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \cdots \qquad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\lim_{x \to 0} \frac{\sin(2x)}{e^{3x} - 1} \sim \frac{2x}{3x} = \frac{2}{3}.$$

$$\lim_{x \to 0} \frac{\sin(2x) - 2x}{(e^{3x} - 1)^3} \sim \frac{-\frac{8x^3}{3!}}{27x^3} \sim \frac{-8/6}{27} = -\frac{4}{81}.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots \qquad e^{3x} = 1 + 3x + \frac{9x^2}{2!} + \cdots$$

$$e^{3x} - 1 = 3x + \frac{9x^2}{2!} + \cdots$$

④

- cheap error bound for e^x. $(x<1)$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \underbrace{\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \cdots}$$

$$\le \underbrace{\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+1)!} + \frac{x^{n+3}}{(n+1)!}}_{\text{geometric series}}$$

$$e^{1/2} = 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \cdots + \frac{(1/2)^n}{n!}$$

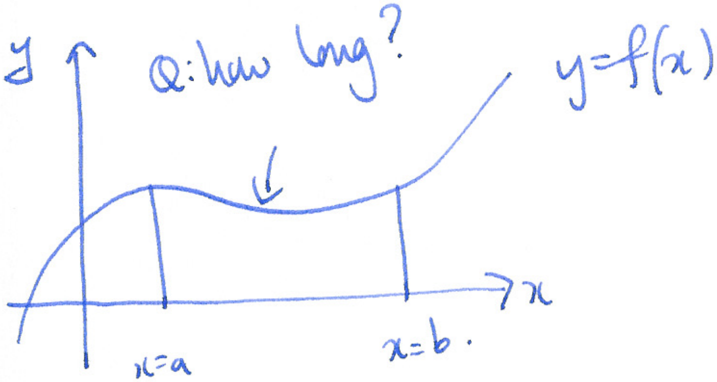
$$\boxed{\frac{x^{n+1}}{(n+1)!}\cdot\frac{1}{1-x}}$$ bound for error term

with error $\frac{(1/2)^{n+1}}{(n+1)!\cancel{n!}} \cdot 2 = \frac{(1/2)^n}{(n+1)!}$

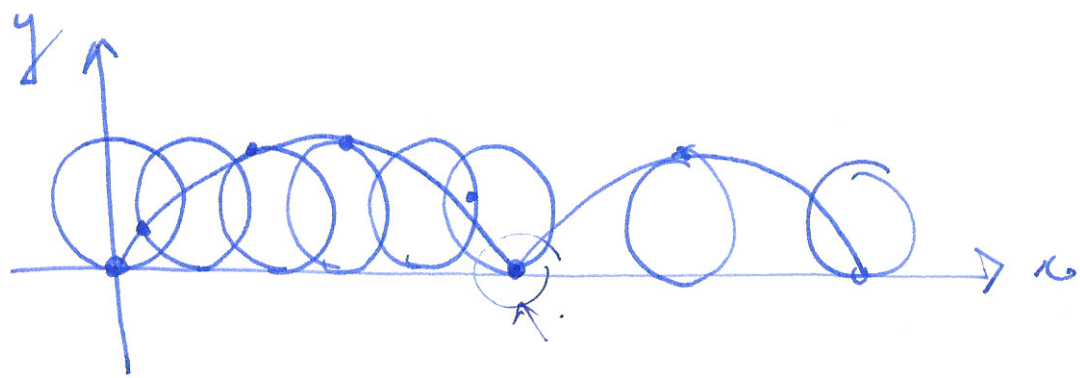
$$e^{1/2} = 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/\cancel{3})^3}{3!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} = \frac{\cancel{14}\ 48+24+6+1}{48} = \frac{79}{48}.$$

$$\left|e^{1/2} - \frac{79}{48}\right| \le \frac{(1/2)^3}{4!} = \frac{1/8}{24} = \frac{1}{193}$$

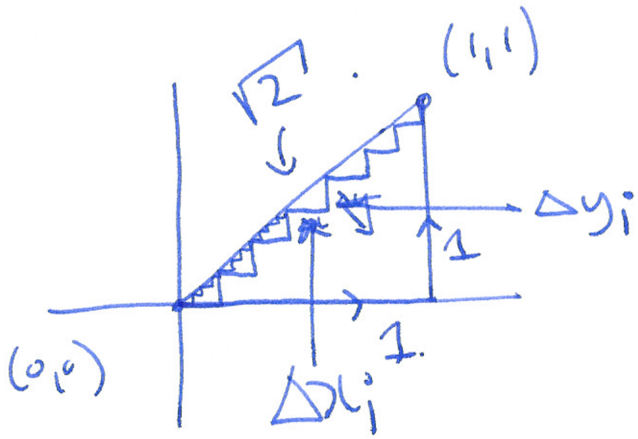
§8.1 Arc length and surface area

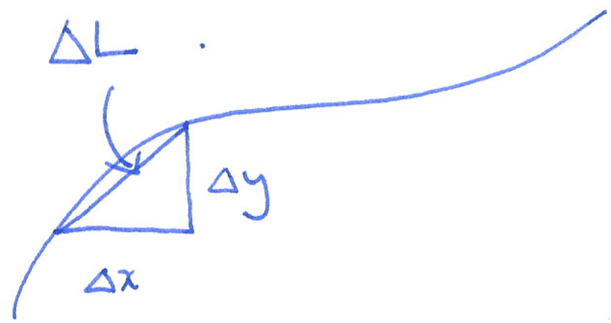


motivation.



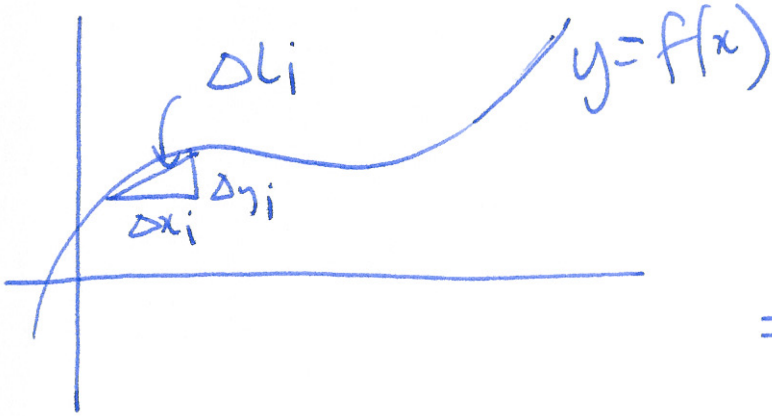
wrong way





$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

$$\text{length} = \sum_{i=1}^{n} \Delta L_i$$



$$\text{length} = \sum_{i=1}^{n} \Delta L_i$$

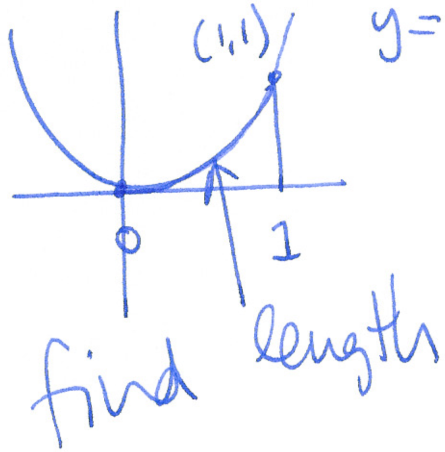
$$= \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}\,.$$

$$= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\ \Delta x_i\,.$$

take limit as $|\Delta x_i| \to 0$.

$$= \boxed{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx} \leftarrow \text{arc length formula.}$$

Example



$y = x^2$.

$$\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx .$$

$$\frac{dy}{dx} = 2x .$$

$$\int_0^1 \sqrt{1 + (2x)^2}\, dx = \int_{x=0}^{x=1} \sqrt{1 + 4x^2}\, dx .$$

$\sin^2\theta + \cos^2\theta = 1$.

$\tan^2\theta + 1 = \sec^2\theta$.

sub $2x = \tan\theta$.

$$2\frac{dx}{d\theta} = \sec^2\theta .$$

$$\int_0^{\tan^{-1}(2)} \sqrt{1 + \tan^2\theta}\, \frac{dx}{d\theta}\, d\theta .$$

$$\int_0^{\tan^{-1}(2)} \sqrt{\underbrace{1+\tan^2\theta}_{\sec^2\theta}} \; \frac{1}{2}\sec^2\theta \, d\theta .$$

$$\int_0^{\tan^{-1}(2)} \frac{1}{2}\sec^3\theta \, d\theta \leftarrow \text{integration by parts.}$$

$$\int uv' \, dx = uv - \int u'v \, dx .$$

$$\frac{1}{2}\int_0^{\tan^{-1}(2)} \underbrace{\sec^2\theta}_{v'} \, \underbrace{\sec\theta}_{u} \, d\theta$$

$u = \sec\theta$ $\qquad$ $v' = \sec^2\theta$

$u' = \sec\theta\tan\theta$ $\qquad$ $v = \tan\theta$.

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec\theta \underbrace{\tan^2\theta}_{\sec^2\theta - 1} \, d\theta .$$

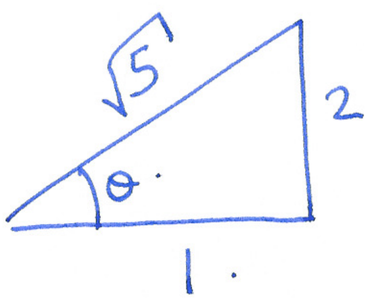
$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta.$$

$$\int \sec^3\theta \, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta \, d\theta.$$

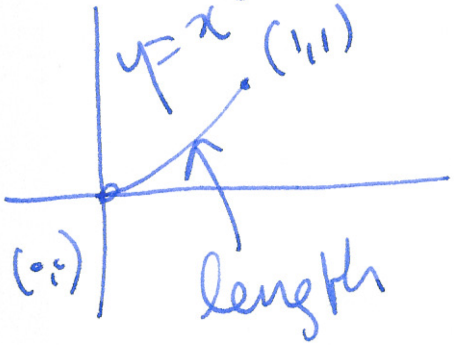
$$2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C.$$

$$\int \sec^3\theta \, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.$$

$$\frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta \, d\theta = \left[\frac{1}{4}\underbrace{\sec\theta}_{\sqrt{5}}\underbrace{\tan\theta}_{2.} + \frac{1}{4}\ln\left|\underbrace{\sec\theta}_{\sqrt{5}.} + \underbrace{\tan\theta}_{2.}\right|\right]_0^{\tan^{-1}(2)}.$$



$\tan\theta = 2.$ $\quad \cos\theta = \dfrac{1}{\sqrt{5}}$ $\quad \sec\theta = \sqrt{5}.$



$$= \frac{1}{4}\left(2\sqrt{5} + \ln\left|\sqrt{5}+2\right|\right).$$

$$\int_{x=0}^{x=1} \sqrt{1+4x^2}\, dx.$$

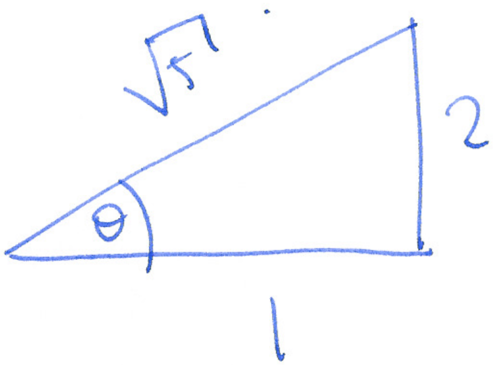
sub

$$2x = \tan\theta.$$

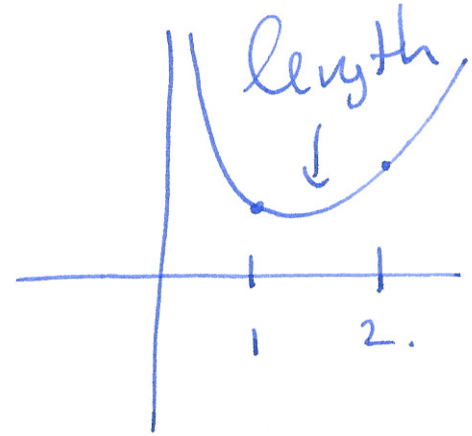
$$2 = \tan\theta.$$

$$\theta = \tan^{-1}(2).$$

$$\int_0^{\tan^{-1}(2)} \underline{\qquad\qquad} \, d\theta.$$



Example $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1,2]$.



arc length $= \displaystyle\int_1^2 \sqrt{1+(f'(x))^2}\, dx$.

$$f'(x) = \frac{1}{4}x^2 - \frac{1}{x^2}$$

$$\int_1^2 \sqrt{1+\left(\frac{1}{4}x^2 - \frac{1}{x^2}\right)^2}\, dx.$$

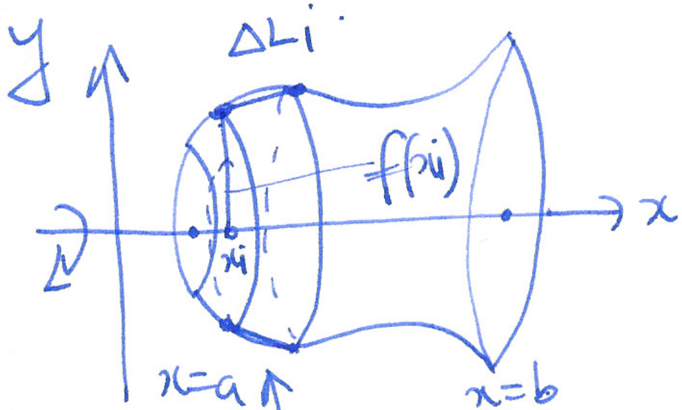
$$\int_1^2 \sqrt{1+\frac{1}{16}x^4 - \frac{1}{2} + \frac{1}{x^4}}\, dx.$$

$$\int_1^2 \sqrt{\frac{1}{16}x^4 + \frac{1}{2} + \frac{1}{x^4}}\,dx = \int_1^2 \sqrt{\left(\frac{1}{4}x^2 + \frac{1}{x^2}\right)^2}\,dx$$

$$= \int_1^2 \frac{1}{4}x^2 + \frac{1}{x^2}\,dx = \left[\frac{1}{12}x^3 - \frac{1}{x}\right]_1^2$$

$$= \frac{\cancel{8}\,4}{6\,\cancel{12}} - \frac{1}{2} - \frac{1}{12} + 1 = \cancel{\frac{4}{3}} = \frac{8-6-1+12}{12} = \frac{13}{12}$$

area for surfaces of revolution.



$y=f(x)$.

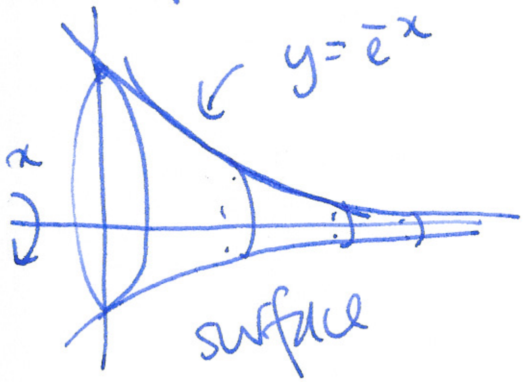
$$\boxed{\text{area} = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx}.$$

$$\sum \text{area of cylinder} \approx \sum \Delta L_i \, 2\pi f(x_i).$$

$$\approx \sum 2\pi f(x_i)\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}.$$

$$\simeq \sum 2\pi f(x_i)\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\;\Delta x_i.$$

Example



$y = e^{-x}$ $\qquad \dfrac{dy}{dx} = -e^{-x}.$

$$\int_0^\infty 2\pi\, e^{-x}\sqrt{1+(-e^{-x})^2}\,dx.$$

$$\int_0^\infty \cancel{2\pi}\; e^{-x}\sqrt{1+e^{-2x}}\,dx.$$

$2\pi.$

try. $u = e^{-x}.$

$$\frac{du}{dx} = -e^{-x}.$$

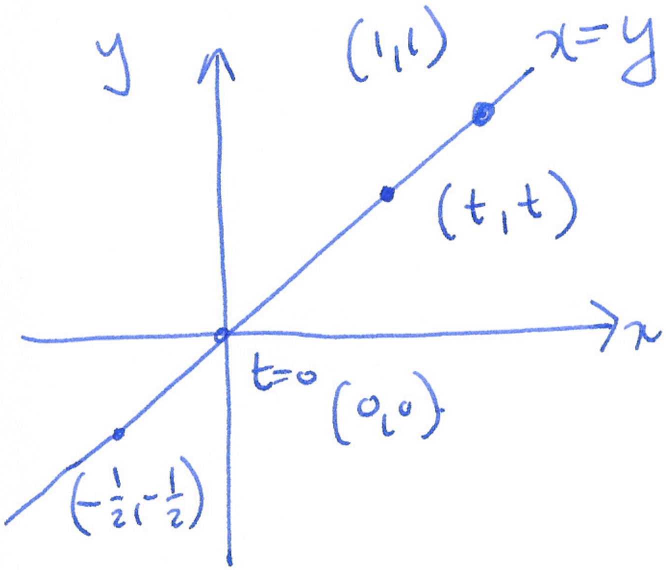
$$2\pi\int \cancel{e^{-x}}\; u\sqrt{1+u^2}\;\frac{dx}{\boxed{du}}\,du.$$

$\cancel{1/-e^{x}}$

$$= -2\pi\int\sqrt{1+u^2}\,du.$$

trig sub.

§11.1 Parametric equations

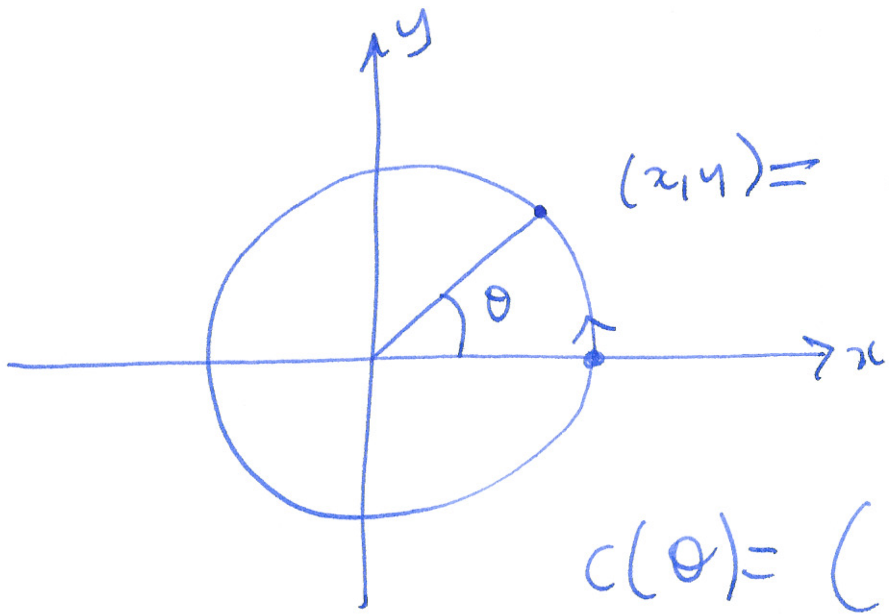


$y = x$.

$c(t) = (t, t)$.

parameterized curve.

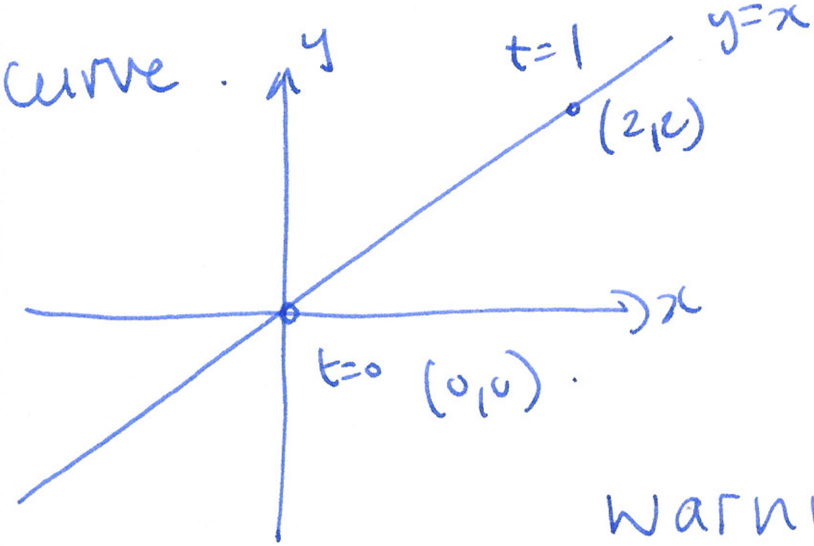
$c(t) = (x(t), y(t))$.



$(x, y) = (\cos\theta, \sin\theta)$.

$c(\theta) = (\cos\theta, \sin\theta)$.

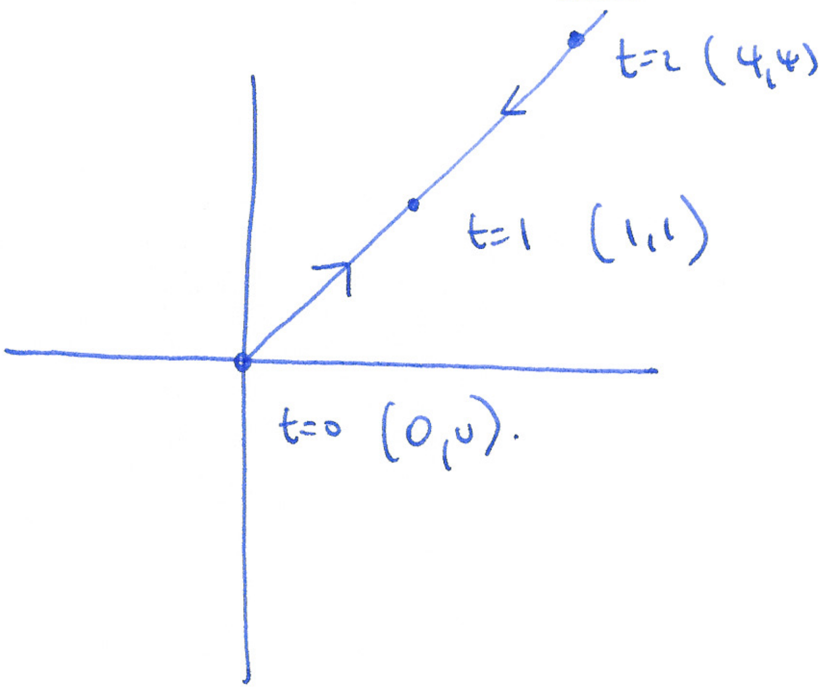
<u>problems</u> many different parameterizations give the same curve.



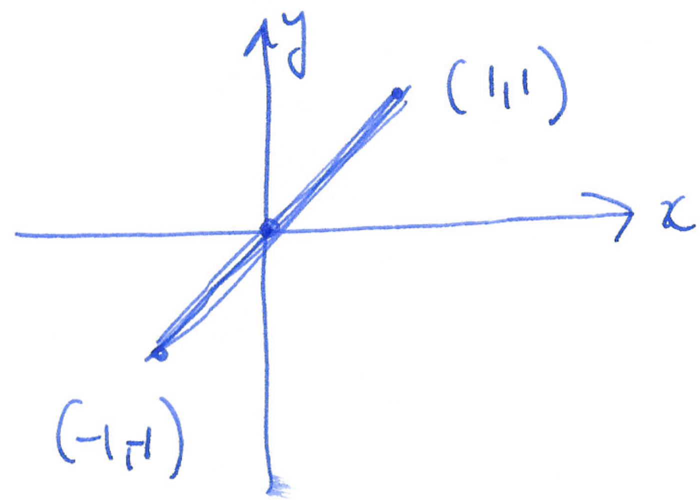
$$c(t) = (2t, 2t).$$

$$c(t) = (t^3, t^3).$$

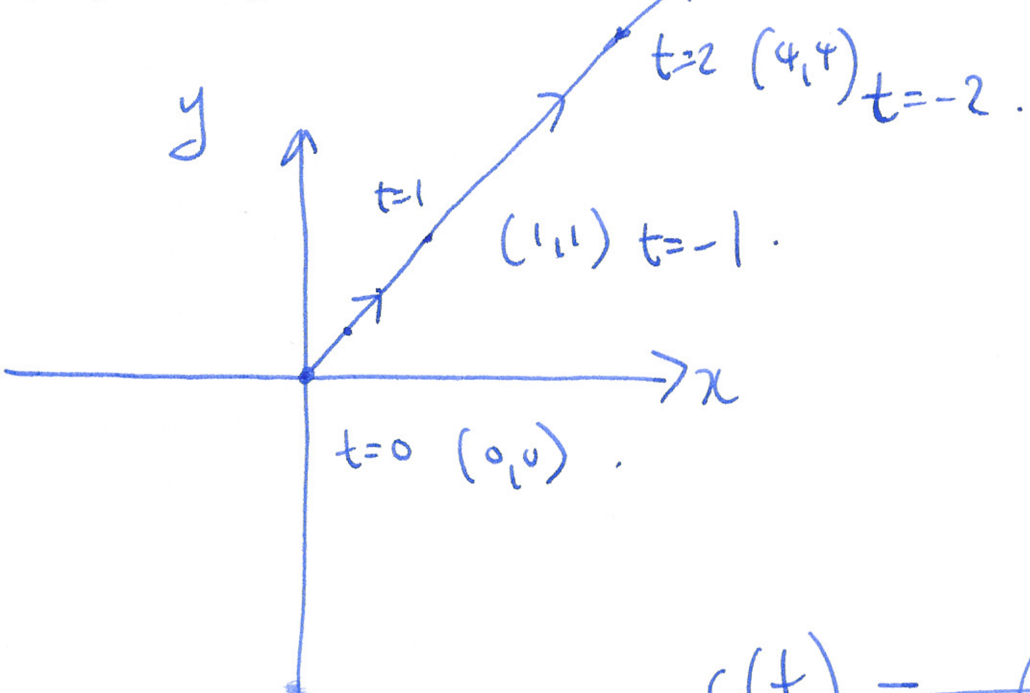
<u>warning</u> $c(t) = (t^2, t^2)$.



Q: $c(t) = (\sin t, \sin t)$.



$c(t) = (x(t), y(t))$.



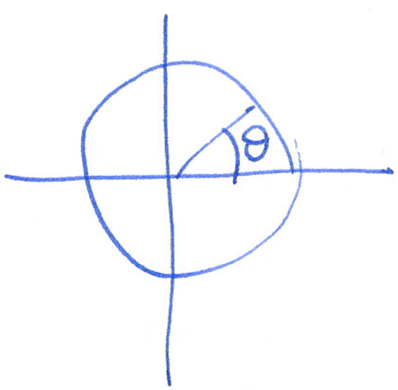
$$c(t) = (\underset{x}{\underset{\parallel}{t^2}}, \underset{y}{\underset{\parallel}{t^2}}).$$

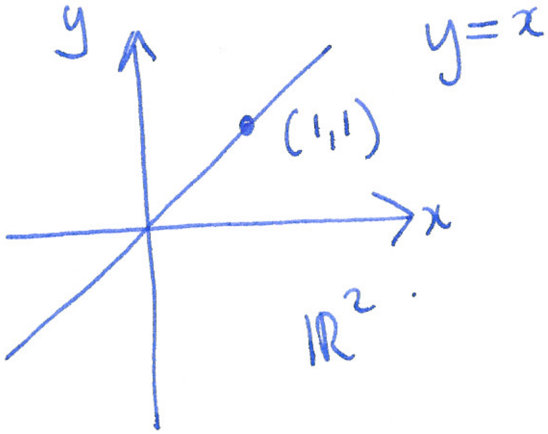
$$\underset{x}{\underset{\parallel}{t^2}} = \underset{y}{\underset{\parallel}{t^2}}$$

$$x = y$$

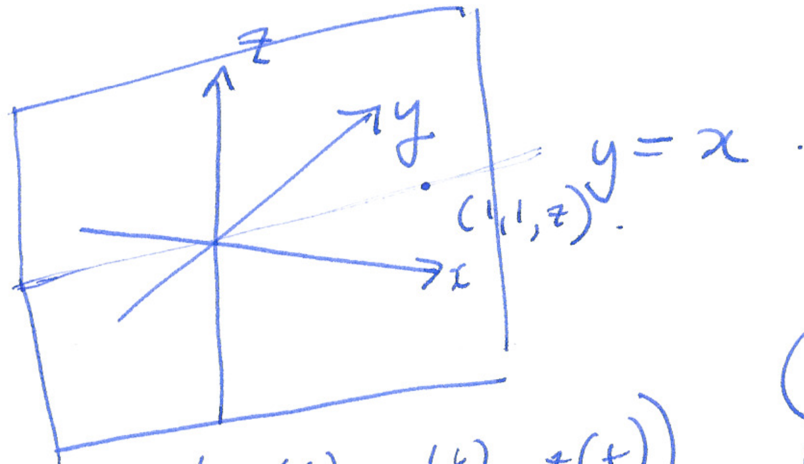
$$\cancel{c(t) = \left(x^2 + \tan x,\ e^x - \ln|x|\right)}.$$

$$c(t) = (\underbrace{t^2 + \tan t}_{x(t)},\ \underbrace{e^t - \ln|t|}_{y(t)}).$$





$c(t) = (t, t)$



$(x(t), y(t), z(t))$.

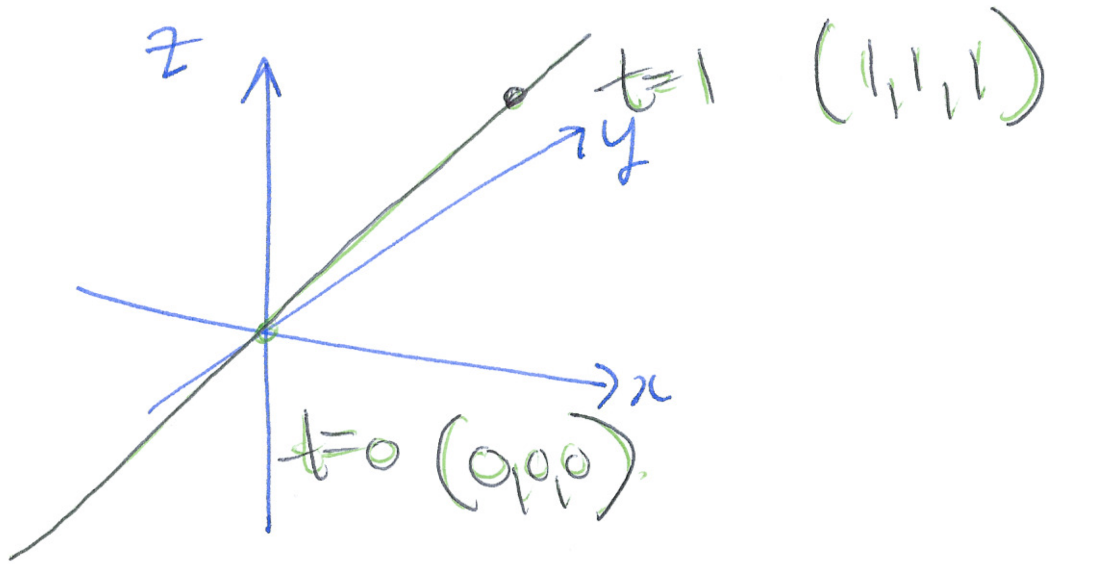
$c(t) = (t, t, t)$.

$y = x$

$(1,1,0)$

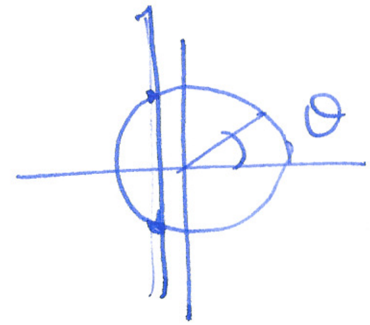
$(1,1,1)$

$(1,1,100)$.



<u>Example</u> 1 describe the parameterized curve

$$c(t) = (\underbrace{t+1}_{x(t)}, \underbrace{t^2+1}_{y(t)}) \quad \text{as} \quad y = f(x).$$



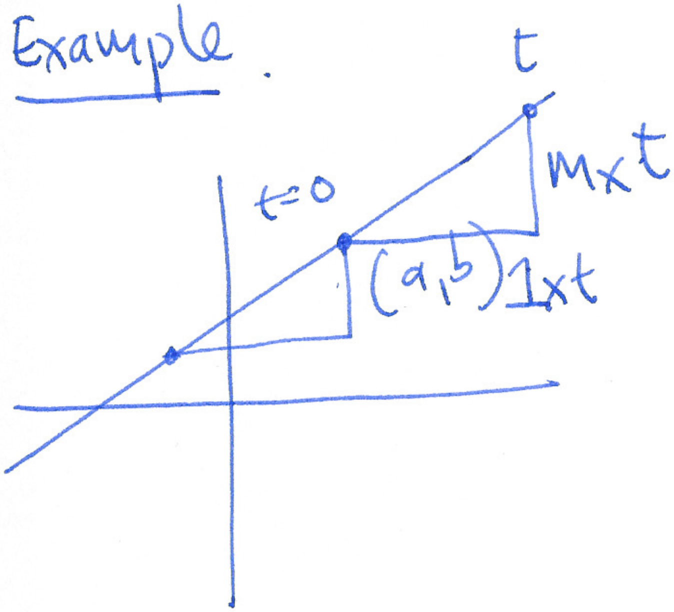
$$\left.\begin{aligned} x &= t+1 \\ y &= t^2+1 \end{aligned}\right\} .$$

$t = x - 1.$

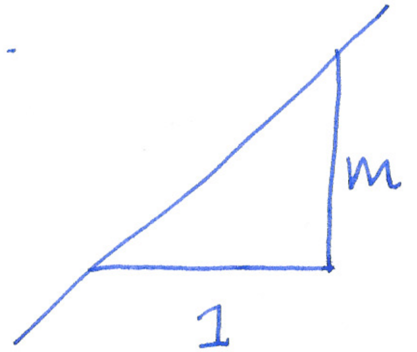
$y = (x-1)^2 + 1.$

$y = x^2 - 2x + 2.$

Example.



find a parameterization for the line through (a,b) with slope m.



$t=0$ $\quad (a,b)$.

$t=1$ $\quad (a+1, b+m)$.

$$c(t) = (a+t, b+mt).$$