10.2 Q6 $\qquad \sum_{n=1}^{\infty} a_n \qquad s_N = 7 - \frac{8}{N^2}$

(1)

a) $\sum_{n=1}^{10} a_n = 7 - \frac{8}{100}$

$\sum_{n=4}^{16} a_n = S_{16} - S_3 = 7 - \frac{8}{16^2} - \left(7 - \frac{8}{9}\right) = \frac{8}{9} - \frac{8}{16^2}.$

b) $a_3 = S_3 - S_2 = 7 - \frac{8}{9} - \left(7 - \frac{8}{4}\right) = \frac{8}{4} - \frac{8}{9} = 2 - \frac{8}{9} = \frac{10}{9}.$

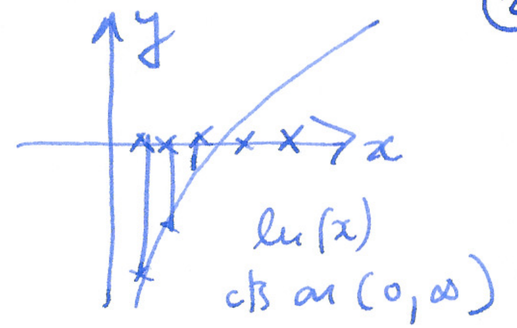
c) $a_n = S_n - S_{n-1} = \left(7 - \frac{8}{n^2}\right) - \left(7 - \frac{8}{(n-1)^2}\right) = \frac{8}{(n-1)^2} - \frac{8}{n^2}.$

d) $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} 7 - \frac{8}{N^2} = 7$

b). $S_3 = a_1 + a_2 + a_3 \qquad S_3 - S_2 = a_3.$

$S_2 = a_1 + a_2$

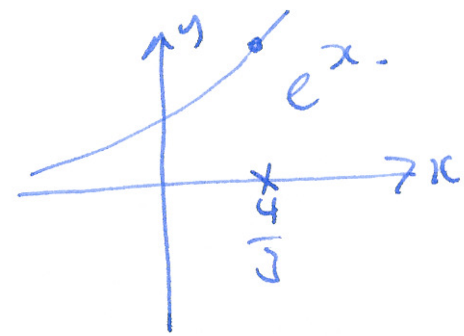
10.1 Q5 (d_n) $d_n = \ln(16^n) - \ln(n!)$.



$$d_n = \ln\left(\frac{16^n}{n!}\right) \rightarrow -\infty. \quad \text{DNE}.$$

$$\lim_{n\to\infty} \frac{16^n}{n!} = 0 \quad \text{recall.} \quad \frac{16 \cdot 16 \cdot \ldots \quad 16}{1 \cdot 2 \cdot \ldots \quad n}.$$

$$= \underbrace{\frac{16}{1}\cdot\frac{16}{2}\cdots\frac{16}{16}}_{A} \cdot \underbrace{\frac{16}{17}\cdot\frac{16}{18}\cdots\frac{16}{n-1}}_{<1} \cdot \frac{16}{n}. \leq A \cdot \frac{16}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$



10.1 Q4. $a_n = e^{4n/(3n+11)}$.

$$\lim_{n\to\infty} \frac{4n}{3n+11} = \lim_{n\to\infty} \frac{4}{3 + 11/n}. = \frac{4}{3}.$$

$$\lim_{n\to\infty} a_n = e^{4/3}.$$

<u>recall</u> $a_n \to L$ as $n \to \infty$. $f(x)$ cts at L

then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$.

<u>recall</u> power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ centered at c.

radius of convergence $R \in [0, \infty]$.

to find R: use ratio test.

<u>Example</u> $\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2x^2 + 3!\,x^3 + 4!\,x^4 + \cdots$ ← centered at 0.

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right|$

$= \lim_{n\to\infty} \left|\frac{\cancel{1}\cdot\cancel{2}\cdots\cancel{n}\cdot(n+1)}{\cancel{1}\cdot\cancel{2}\cdots\cancel{n}}\,x\right| = \lim_{n\to\infty} (n+1)|x| = \infty$ if $|x| \neq 0$.

> 1. doesn't converge.

so $\boxed{R = 0}$.

Thm Suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence $R > 0$ then $F(x)$ is differentiable on $(a-R, a+R)$ and furthermore

$$\left.\begin{aligned} F'(x) &= \sum_{n=1}^{\infty} a_n\, n\, (x-a)^{n-1} \\ \int F(x)\,dx &= \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C. \end{aligned}\right\}$$ same radius of convergence.

Examples (1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\qquad a_n = \frac{x^n}{n!}$

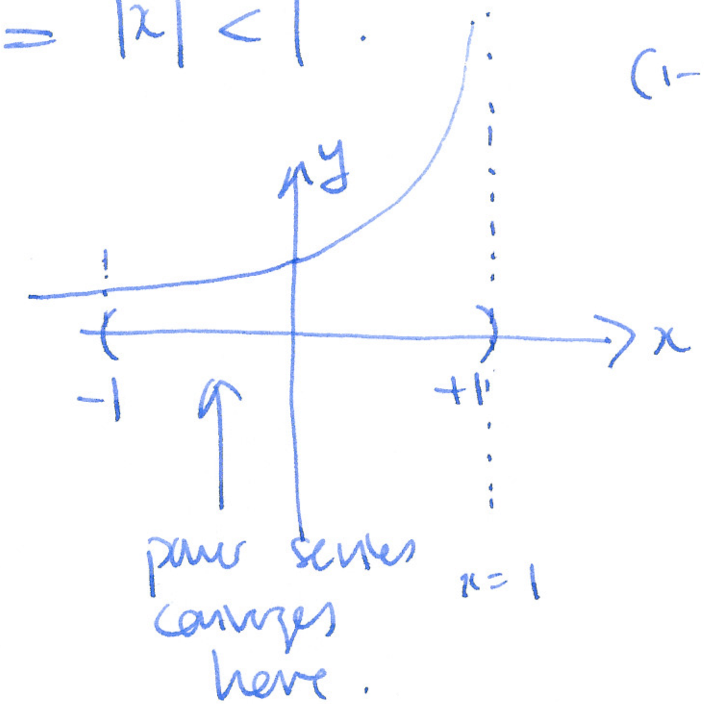
$(e^x)' = 1 + x + \frac{x^2}{2!} + \cdots$ $\qquad \frac{n x^{n-1}}{n!} = \frac{x^{n-1}}{(n-1)!}$

(2) $\underbrace{1+x+x^2+x^3+x^4+\cdots}_{|x|<1.} = \sum_{n=0}^{\infty} x^n = \underbrace{\frac{1}{1-x}}_{x\neq 1.}, \quad |x|<1$

$c+cr+cr^2+cr^3+\cdots = \frac{c}{1-r}$

check : use ratio test $\lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}|x|.$

$= |x|<1.$



$(1-x)^{-1} = \frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots$

$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+\cdots$

$\int\frac{1}{1-x}dx = c+x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots$

$-\ln|1-x| = x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\cdots$

$$1-x^2+x^4-x^6+\cdots = \frac{1}{1+x^2}. \qquad r=-x^2$$

$$c+cr+cr^2+cr^3+\cdots = \frac{c}{1-r}.$$

non-example.

not geometric.

$$\boxed{\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots}$$

integrate:

$$\boxed{\tan^{-1}(x) = \underbrace{x-\frac{1}{3}x^3}_{-\frac{1}{3}x^2}+\underbrace{\frac{1}{5}x^5-\frac{1}{7}x^7}_{\to -\frac{5}{7}x^2}+\cdots}$$

geometric series: $c+cr+cr^2+cr^3+\cdots$.

$$S_N = c+cr+cr^2+\cdots+cr^N.$$

$$rS_N = cr+cr^2+\cdots+cr^N+cr^{N+1}$$

$$S_N-rS_N = c-cr^{N+1}$$

$$S_N(1-r) = c(1-r^{N+1}) \longrightarrow S_N = \frac{c}{1-r}(1-\underbrace{r^{N+1}}_{\to 0.}).$$

$$c+cr+\cdots = \frac{c}{1-r}. \qquad |r|<1.$$

§10.7 Taylor series

suppose $f(x)$ has a power series centered at c, so

$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ Q. what are the a_i?

A. $\boxed{f(c) = a_0}$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

$\boxed{f'(c) = a_1}$

$f''(x) = 2a_2 + 3!(x-c) + 4.3.(x-c)^2 + \cdots$

$\boxed{f''(c) = 2a_2}$ $a_2 = \dfrac{f''(c)}{2}$.

Fact. $\boxed{a_n = \dfrac{f^{(n)}(c)}{n!}}$

Examples ① $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

$f(x) = (1-x)^{-1}$ $\quad f(0) = 1 \quad a_0 = f(0) = 1.$

$f'(x) = +(1-x)^{-2}$ $\quad f'(0) = 1 \quad a_1 = \frac{f'(0)}{1} = 1.$

$f''(x) = 2(1-x)^{-3}$ $\quad f''(0) = 2 \quad a_2 = \frac{f''(0)}{2!} = \frac{2}{2} = 1.$

$f^{(3)}(x) = 2.3(1-x)^{-4}$ $\quad f^{(3)}(0) = 3! \quad a_3 = \frac{f^{(3)}(0)}{3!} = \frac{3!}{3!} = 1.$

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

Example e^x

$f(x) = e^x \qquad f(0) = 1 \qquad a_0 = f(0) = 1.$

$f'(x) = e^x \qquad f'(0) = 1 \qquad a_1 = \frac{f'(0)}{1!} = 1.$

$f''(x) = e^x \qquad f''(0) = 1. \qquad a_2 = \frac{f''(0)}{2!} = \frac{1}{2!}$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1 \qquad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

center $x=0$. (10)

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$f(0) = a_0$$

$a_n x^n$.

$a_n n x^{n-1}$

$a_n n(n-1) x^{n-2}$

$a_n n(n-1)(n-2) x^{n-3}$

$\vdots$

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$f'(0) = a_1$$

$$f''(0) = 2a_2 + 3.2.a_3 x + 4.3 a_4 x^2 + \cdots$$

$$f''(0) = 2a_2.$$

$$f^{(n)}(x) = a_n \, n! + \cdots$$

$$f^{(3)}(x) = 3.2.a_3 + 4.3.2.x + \cdots$$

$$f^{(3)}(0) = 3! a_3.$$

$$f^{(n)}(0) = n! \, a_n.$$

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

$$f(x) = \sqrt{x^2-1} = (x^2-1)^{1/2} = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

centered at 2.

$$f(x) = (x^2-1)^{1/2} \qquad f(2) = \sqrt{3}.$$

$$f'(x) = \frac{1}{\cancel{2}}(x^2-1)^{-1/2} \cdot \cancel{2}x \qquad f'(2) = \frac{2}{\sqrt{3}}.$$

$$f''(x) = -\frac{1}{\cancel{2}}(x^2-1)^{-3/2} \cdot \cancel{2}x \cdot x + (x^2-1)^{-1/2}. \qquad f^{(2)}(2) = -\frac{4}{\sqrt{3}} + \frac{1}{\sqrt{3}} = -\frac{3}{\sqrt{3}}.$$

$$f^{(3)}(x) = \quad \cdots\cdots$$

$$\sqrt{x^2-1} = \cancel{\sqrt{3} + \frac{2}{\sqrt{3}}x - \frac{3}{\sqrt{3}}x^2}$$

$$= \sqrt{3} + \frac{2}{\sqrt{3}}(x-2) \cancel{+} - \frac{3}{\sqrt{3}}(x-2)^2 + \cdots$$

WW 10.2 Q4 $\quad \displaystyle\sum_{n=0}^{\infty} 3\left(\frac{\pi}{\ln(2)}\right)^{n}$ (12)

$$3 + 3\frac{\pi}{\ln(2)} + 3\cdot\left(\frac{\pi}{\ln(2)}\right)^2 + 3\left(\frac{\pi}{\ln(2)}\right)^3 + \cdots$$ doesn't converge.

$c = 3$

$$c + cr + cr^2 + cr^3$$

$r = \dfrac{\pi}{\ln(2)}$.

$\pi \sim 3$

$\ln(2) < 1$. $\quad \dfrac{\pi}{\ln(2)} > 1$.

WW 10.2 Q 3. $\displaystyle\sum_{n=3}^{\infty} \frac{6.(-7)^n}{14^n}$. ~~ANS~~. $\displaystyle\sum_{n=3}^{\infty} \frac{6.(-1)^n}{2^n}$. (13)

$$= -\frac{6}{8} + \frac{6}{16} - \frac{6}{32} + \cdots\cdots \qquad c = -6/8.$$

$$r = -1/2. \quad \boxed{|r|<1} \ \checkmark.$$

$$c + cr + cr^2 + \cdots\cdots$$

$$\frac{c}{1-r} = \frac{-6/8}{1+1/2} = \frac{-3/4}{3/2} = -\frac{1}{2}.$$

$$\boxed{\frac{(-7)^n}{(14)^n}} = \frac{(-1)^n(7)^n}{(2\times7)^n} = \frac{(-1)^n\cancel{7^n}}{2^n\cancel{7^n}} = \frac{(-1)^n}{2^n} = \boxed{\left(-\frac{1}{2}\right)^n}.$$

$r = -2$. $|r| = 2 > 1$.

we can use power series to solve differential equations.

$x^2 = 4$. $\quad y' = y \quad \frac{dy}{dx} = y \quad y = Ae^x$.

look for a solution $y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$a_1 = a_0$

$2a_2 = a_1 = a_0 \qquad a_2 = \frac{a_0}{2}$.

$3a_3 = a_2 = \frac{a_0}{2} \qquad a_3 = \frac{a_0}{3!}$

solution $\quad a_0 + a_0 x + \frac{a_0}{2} x^2 + \frac{a_0}{3!} x^3 + \cdots$

$a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)$.

e^x.

$y'' + y' + y = 0$. look for a solution: $y(0) = 1 \quad y'(0) = 1$. (15)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$a_0 = 1$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$a_1 = 1.$$

$$y'' = 2a_2 + 6a_3 \cdot x + \cdots$$

$1 + 1 + 2a_2 = 0$. $\quad a_2 = -1$.

$1 + 2(-1) + 6a_3 = 0 \quad a_3 = +1/6$.

soln. $\quad 1 + x - x^2 + \frac{1}{6}x^3 + \cdots$

useful facts when the Taylor series converges, we can

- differentiate
- integrate
- multiply $\leftarrow xe^x$
- substitute $\leftarrow e^{x^2}$

Example

$$f(x) = xe^x. \qquad f(0) = 0$$

$$f'(x) = e^x + xe^x. \qquad f'(0) = 1$$

$$f''(x) = e^x + e^x + xe^x = 2e^x + xe^x \qquad f''(0) = 2 \quad \dots$$

$$x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$$

<u>Example</u> $f(x) = e^{x^2}$.

$f(0) = 1$.

$f'(x) = e^{x^2} \cdot 2x$.

$f'(0) = 0$

$f''(x) = e^{x^2} \cdot 4x^2 + e^{x^2} \cdot 2$.

$f''(0) = 2$

.....

$f^{(3)}(x) = \ldots\ldots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

<u>Example</u> $f(x) = e^{-x^2}$

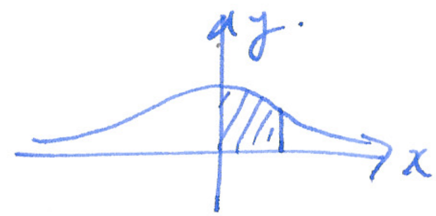
$$f'(x) = e^{-x^2} \cdot -2x$$

$$f''(x) = e^{-x^2} \cdot 4x^2 - 2e^{-x^2}$$

$$\vdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$



$$\int e^{-x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots$$

<u>Fact</u> all this works for complex numbers $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \leftarrow \text{centered at } 0$$

$$e^{x+1} = 1 + (x+1) + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \cdots \quad \leftarrow \text{centered at } -1.$$

$$e^{x+1} = e \cdot e^x = e + ex + e\frac{x^2}{2!} + \cdots$$

$$(x+1)e^x = (x+1) + (x+1)x + (x+1)\frac{x^2}{2!} + (x+1)\frac{x^3}{3!} + \cdots$$

power series $\displaystyle\sum_{n=0}^{\infty} a_n (x-c)^n.$

$$\begin{aligned} = \; & 1 + x \\ & \quad x + x^2 \\ & \qquad \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned} \qquad = 1 + 2x + \left(1 + \frac{1}{2!}\right)x^2 + \cdots$$

$$f(x) = e^x \sin x.$$

$$= \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdot\right).$$

$$\begin{array}{lllll} x & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} \\ & +x^2 & & -\frac{x^4}{3!} & & +\frac{x^6}{5!} \\ & & +\frac{x^3}{2!} & & -\frac{x^5}{2!3!} & & + \cdots \end{array}$$

$$x + x^2 + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \cdots\cdots$$