last time : series $\longleftrightarrow$ infinite sums.

special case: positive series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$

$a_n \geq 0$.

Examples. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$. $c + cr + cr^2 + cr^3 + \cdots = \frac{c}{1-r}$

$|r| < 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\leftarrow$ converges if $p > 1$
diverges if $p \leq 1$.

Thm Comparison test suppose $0 \leq a_n \leq b_n$ for all $n \geq M$.

then ① if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

② if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges.

Example $\sum_{n=1}^{\infty} 2^{-n^2} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \frac{1}{2^{16}} + \dots$

↑ compare to geometric series

$n^2 > n$ $\quad -n^2 < -n$ $\quad \frac{1}{2^{n^2}} < \frac{1}{2^n}$ $\quad a_n = \frac{1}{2^{n^2}}$ $\quad b_n = \frac{1}{2^n}$.

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ geometric series, converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^{n^2}}$ converges

Thm Limit comparison test a_n, b_n positive series.

suppose $\lim_{n \to \infty} \frac{a_n}{b_n} = L < \infty$ exists.

→ iff $\Leftrightarrow$ if and only if

$\leftrightarrow$ $\Longleftrightarrow$

then ① if $L > 0$ $\sum_{n=1}^{\infty} a_n$ converges $\boxed{\text{iff}}$ $\sum_{n=1}^{\infty} b_n$ converges.

② if $L = 0$ $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof (sketch) case $L > 0$, $\frac{a_n}{b_n} \to L$ so $0 < \frac{a_n}{b_n} < R_1$.

for some $R > L$, so $0 < a_n < R b_n$.

comparison test $\sum b_n$ converges $\Rightarrow$ $R \sum b_n$ converges. $\Rightarrow$ $\sum a_n$ converge.

$\frac{b_n}{a_n} \to \frac{1}{L}$ so there is R_2' s.t. $0 < \frac{b_n}{a_n} < R'$

so $0 < b_n < R' a_n$ $\sum a_n$ converge, $R' \sum a_n$ converge $\Rightarrow$ $\sum b_n$ converge. □

Example show $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1} = a_n$. converges. (intuition $a_n \sim \frac{1}{n^2}$).

compare with $b_n = \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^4-n-1}{n^2}$

$= \lim_{n\to\infty} \frac{n^4-n-1}{n^4} = \lim_{n\to\infty} 1 - \frac{1}{n^3} - \frac{1}{n^4} = 1$ $0 < \quad < \infty$.

$\sum \frac{1}{n^2}$ converges. ⊛ $\Rightarrow$ $\sum \frac{n^2}{n^4-n-1}$ converges.

⊛ by limit comparison test

setup $\sum a_n$ $\sum b_n$ positive. $a_n \geq 0$ $b_n \geq 0$.

limit comparison test: $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$.

$a_n = \frac{n^2}{n^4-n-1}$ $b_n = \frac{1}{n^2}$

know $\sum b_n$ converge (p-series) $\overset{LCT}{\Longrightarrow}$ $\sum a_n$ converge.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^4-n-1} \cdot n^2 = \lim_{n\to\infty} \frac{n^4}{n^4-n-1} = 1.$$

$$a_n = \frac{n^2}{n^4-n-1}. = \frac{1}{n^2-\frac{1}{n}-\frac{1}{n^2}} \sim \frac{1}{n^2}.$$

Example does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$ compare with $b_n = \frac{1}{n}$.

a_n

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2-9}} \cdot n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-9}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-9/n^2}}$$

$= 1$

$0 < 1 < \infty$ limit comparison test $\sum a_n$ converges $\Leftrightarrow \sum b_n$ converges.

$\sum_{n=4}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$ diverges.

§10.4 Absolute and conditional convergence

Q $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. ⊛ .

Defn A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

⊛ is absolutely convergent. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ← p-series $p > 1$.

Example $1 - \frac{1}{2} + \frac{1}{3} \cancel{+} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ ← not absolutely convergent

$\sum \frac{1}{n}$ diverges.

Thm absolute convergence $\Rightarrow$ convergence.

Thm absolute convergence ⇒ convergence.

$$\sum_{n=1}^{\infty} |a_n| \Rightarrow \sum_{n=1}^{\infty} a_n$$

Proof $0 \leqslant a_n + |a_n| \leqslant 2|a_n|$

assume absolute convergence $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} 2|a_n|$ converges.

$\Rightarrow$ $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges by the comparison test.

so $\sum_{n=1}^{\infty} a_n + |a_n| - |a_n| = \sum_{n=1}^{\infty} a_n + |a_n| - \sum_{n=1}^{\infty} |a_n|$.

converges. $\Leftarrow$ converges. converges.

↑

$\sum_{n=1}^{\infty} a_n$ converges. $\square$.

Q: what about $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$? $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ $\circledast$.

Defn $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge.

Thm (Alternating series test) Let a_n positive decreasing sequence, $a_n \to 0$ Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges to s. Furthermore $0 \leq s \leq a_1$ and $s_{2n} \leq s \leq s_{2n+1}$ for all n.

Example. $\circledast$ $\sum \frac{(-1)^{n+1}}{n}$ converges. (to $\ln(2)$).

<u>Proof</u>. alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$. <u>$a_i$ decreasing</u>.

even partial sums: $s_{2n} = \underbrace{a_1 - a_2}_{\geqslant 0} + \underbrace{a_3 - a_4}_{\geqslant 0} + \cdots + \underbrace{a_{2n-1} - a_{2n}}_{\geqslant 0}$.

↑ positive increasing ~~series~~. sequence.

odd partial sums: $s_{2n+1} = a_1 - \underbrace{(a_2 - a_3)}_{\geqslant 0} - \underbrace{(a_4 - a_5)}_{\geqslant 0} \cdots - \underbrace{(a_{2n} - a_{2n+1})}_{\geqslant 0}$

↑ decreasing sequence.

Furthermore: $s_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2n}$.

so $s_{2n} \leqslant a_1$ for all n.

so s_{2n} positive increasing sequence, bounded above

$\Rightarrow$ converges. so $\lim\limits_{n \to \infty} s_{2n}$ exists.

similarly $\lim_{n\to\infty} s_{2n+1}$ exists. $s_{2n+1} = \underset{\geq 0}{(a_1-a_2)} + \underset{\geq 0}{(a_3-a_4)} + \cdots + \underset{\geq 0}{(a_{2n-1}-a_{2n})} + a_{2n+1}$

$\geq (a_1-a_2)$.

finally $$\underset{\text{exists}}{\lim_{n\to\infty} s_{2n+1} - s_{2n}} \Leftarrow \underset{\text{exists}}{\lim_{n\to\infty} s_{2n+1}} - \underset{\text{exists}}{\lim_{n\to\infty} s_{2n}}$$

$\lim_{n\to\infty} a_{2n+1} = 0$ (as $a_n \to 0$).

$\Rightarrow$ $\lim_{n\to\infty} s_n$ exists, i.e. $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ exists. $\square$.

Example show $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

use : alternating series test $a_n = \frac{1}{n}$

positive ✓.

decreasing

$a_{n+1} < a_n$.

$\frac{1}{n+1} < \frac{1}{n}$. ✓

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓.

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

so $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges ✓.

§10.5 Ratio and root tests.

Fact $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ Q: how do we know this converges?

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

(e.g. comparison test $n! = 1.2.3 \ldots (n-1).n. > (n-1)^2$

$\frac{1}{n!} < \frac{1}{(n-1)^2}$ $\sum \frac{1}{(n-1)^2}$ converges $\Rightarrow$ $\sum \frac{1}{n!}$ converge).

Fact: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$. ← Stirling's Approx.

Thm Ratio test : (a_n) sequence and suppose

$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then

① if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

② if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

③ if $\rho = 1$ no information.

Proof if $\rho < 1$ then there is a number r $\rho < r < 1$.

and a number N s.t. $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq N$.

so $|a_{N+1}| < r|a_N|$.

$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$. etc.

so $\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| r^n \leq \frac{|a_N|}{1-r}$ so converges by comparison w/ geometric series.

if $\rho > 1$ choose $\rho > r > 1$ and.

~~$\frac{|a_{n+1}|}{|a_N|} > r$~~ $\frac{|a_{n+1}|}{|a_n|} > r$ for all $n \geq N$. so $a_n \not\to 0$

$\Rightarrow \sum a_n$ diverges $\square$.

<u>**Example**</u> ① show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{\cancel{1}.\cancel{2}.\cancel{3}.\ldots \cancel{n}.}{\cancel{1}.\cancel{2}.\cancel{3}.\ldots \cancel{n}.(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

ratio test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

② show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

use ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \Big/ \frac{n^3}{3^n} = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \frac{1}{3}$.

$\lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$ ratio test $\Rightarrow \sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges.

③ Bad example $\sum_{n=1}^{\infty} \frac{1}{n^2}$ apply ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} = 1$$

no information.

Thm Root test (a_n) sequence, suppose that

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists. then

① if $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

② if $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges

③ if $L = 1$ no information.

Example $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$ $a_n = \left(\frac{n}{2n+3}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{2n+3}\right)^n} = \lim_{n\to\infty} \frac{n}{2n+3} = \lim_{n\to\infty} \frac{1}{2+3/n} = \frac{1}{2} < 1$$

converges by ~~ratio~~ test.
root.

§ 10.6 Power series

Defn A power series centered at a is an infinite sum of the form $\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots$

note · if this series converges, defines a function of x, $f(x)$.

· this always converges for $x=a$. $f(a) = a_0$.

Example. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ ← does this converge?

$a_n = \frac{x^n}{n!}$

apply ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right|$.

$= \lim_{n\to\infty} |x^{a}| \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1$ converges by ratio test for all x!

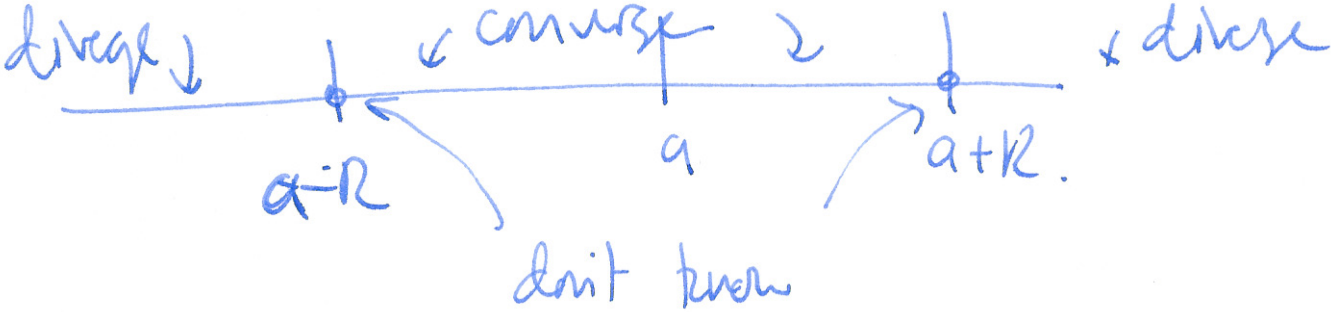
Thm Radius of convergence $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$.

① $F(x)$ converges only for $x=a$ $(R=0)$. radius of convergence.

or ② $F(x)$ converges for all x. $(R=\infty)$.

or ③ ~~$F(x)$~~ there is an $0<R<\infty$ s.t. $F(x)$ converges for all $|x-a|<R$ and diverges for $|x-a|>R$. May or may not converge for $|x-a|=R$ (~~x~~ $a+R$, $a-R$).

R is called the radius of convergence.



Example for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/2^{n+1}}{x^n/2^n} \right| = \xcancel{\lim_{n\to\infty} \frac{|x|}{2}}.$

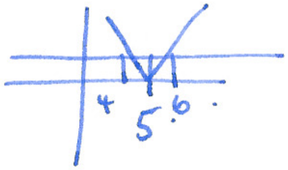
$$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{2^n}{2^{n+1}} \right| = \lim_{n\to\infty} \frac{|x|}{2} = \frac{|x|}{2} < 1$$ then converges.

$|x| < 2$. so radius of convergence is $R = 2$.

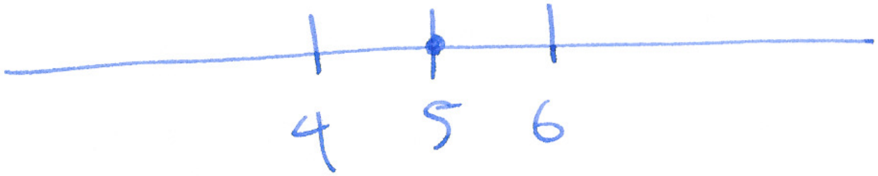
interval of convergence $(-2, 2)$.

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-5)^n$ ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$

$$= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{n+1}(x-5)^{n+1}}{\frac{(-1)^n}{n}(x-5)^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1}(x-5) \right| = |x-5| \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{1.}$$



$= |x-5|$ converges if $|x-\underset{\substack{\uparrow\\ \text{center}}}{5}| < \underset{\substack{\uparrow\\ \text{radius}}}{1}$ radius of convergence is 1.



interval of convergence is ~~(4,6)~~

(4,6].

Example $\sum_{n=0}^{\infty} \frac{x^n}{n!} \leftarrow R=\infty.$

$\sum_{n=0}^{\infty} n!\,x^n. \leftarrow$ show $R=0.$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n+1}(x-5)^{n+1}}{\frac{(-1)^{n}}{n\cancel{+1}}(x-5)^{n}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n+1}(x-5)^{n+1}}{\frac{1}{n}(x-5)^{n}}\right|.$$

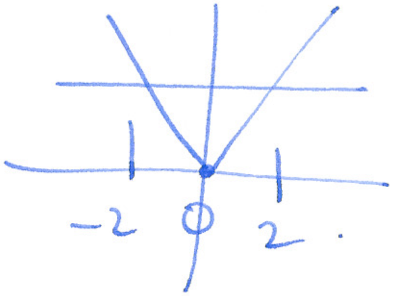
$$\cancel{\lim_{n\to\infty}\left|\frac{n}{n+1}(x-5)\right|} \qquad \lim_{n\to\infty}\left|\frac{1/n+1}{1/n}\,\frac{(x-5)^{n+1}}{(x-5)^{n}}\right|.$$

$$\lim_{n\to\infty}\left|\frac{n}{n+1}(x-5)\right| = |x-5|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|.$$

$$= |x-5|\lim_{n\to\infty}\left|\frac{1}{1+1/n}\right| = |x-5|.$$

$$\lim_{n\to\infty} \left| \quad \right| = \frac{|x|}{2} < 1 .$$

$$|x| < 2 .$$



$(-2, 2)$ $R = 2$.

$|x-5| < 1$.

WARNING

$|x-5| < 1$.

$\Rightarrow |x| < 6$. ✗✗ .