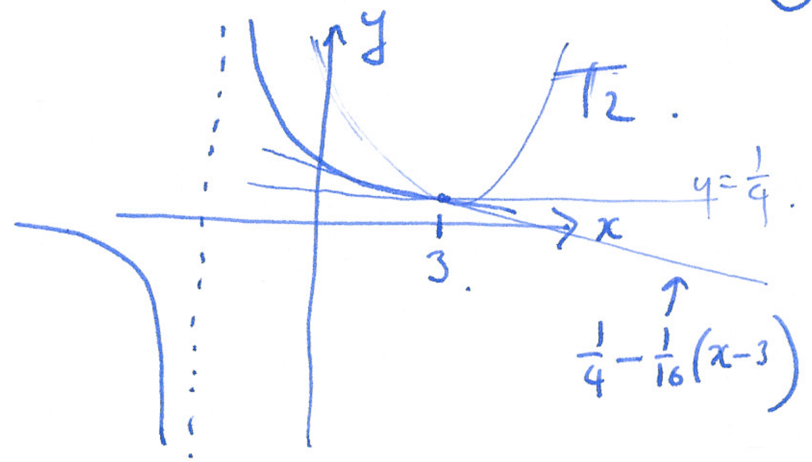
$$f(x) = \frac{1}{1+x} \qquad T_2, T_3 \quad \text{at } a = 3.$$



$$T_3 = \underbrace{a_0}_{f(3)} + \underbrace{a_1}_{f'(3)}(x-3) + \underbrace{a_2}_{\frac{f''(3)}{2!}}(x-3)^2 + \underbrace{a_3}_{\frac{f^{(3)}(3)}{3!}}(x-3)^3.$$

$$f(x) = (1+x)^{-1} \qquad f(3) = 1/4.$$

$$f'(x) = -(1+x)^{-2} \qquad f'(3) = -1/16.$$

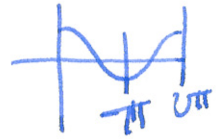
$$f''(x) = 2(1+x)^{-3}. \qquad f''(3) = 2/64 = 1/32.$$

$$f^{(3)}(x) = -6(1+x)^{-4} \qquad f^{(3)}(3) = -6/4^4 = -3/128$$

$$T_2 = \frac{1}{4} - \frac{1}{16}(x-3) + \frac{1}{64}(x-3)^2.$$

$$T_3(x) = \frac{1}{4} - \frac{1}{16}(x-3) + \frac{1}{64}(x-3)^2 - \frac{3}{128.6}(x-3)^2$$

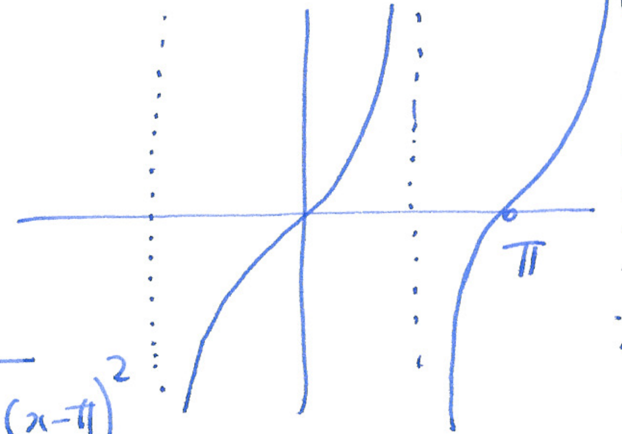
<u>Q3</u> $f(x) = \tan(x)$ T_2, T_3 $x = \pi$.



$f'(x) = \sec^2(x)$.

$f''(x) = 2\sec(x) \cdot \sec(x)\tan(x) = 2\sec^2(x)\tan(x)$

$f^{(3)}(x) = 4\sec(x) \cdot \sec(x)\tan(x) \cdot \tan(x) + 2\sec^2(x) \cdot \sec^2(x)$



$a_0 + a_1(x-\pi) + a_2(x-\pi)^2$

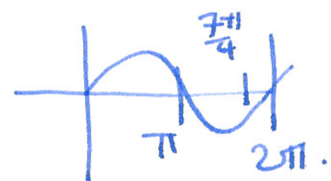
$f(\pi) = 0$ $T_2(x) = 0 + 1(x-\pi) + 0$

$f'(\pi) = 1$

$f''(\pi) = 0$ $T_3(x) = 0 + (x-\pi) + \frac{2}{3!}(x-\pi)^3$.

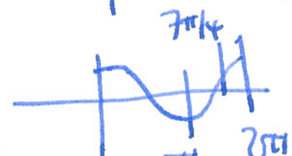
$f^{(3)}(\pi) = 2$ $= (x-\pi) + \frac{1}{3}(x-\pi)^3$

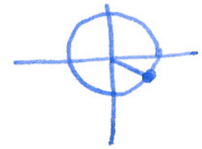
$a = \frac{7\pi}{4}$. $f\left(\frac{7\pi}{4}\right) = -1$ $\sin\left(\frac{7\pi}{4}\right) =$



$-\sin(\pi/4) = -\frac{1}{\sqrt{2}}$.

$f'\left(\frac{7\pi}{4}\right) = 2$ $\cos\left(\frac{7\pi}{4}\right)$. $\cos(7\pi/4) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.



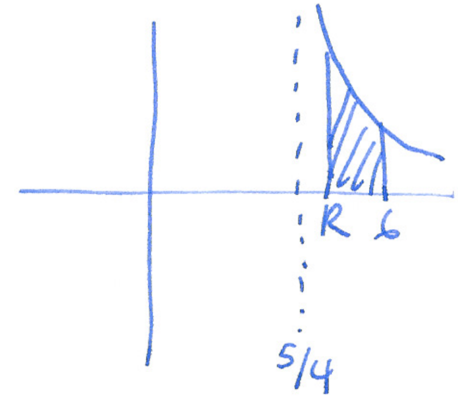


$f''\left(\frac{7\pi}{4}\right) = -4$ $\tan\left(\frac{7\pi}{4}\right) = -1$.

$f^{(3)}\left(\frac{7\pi}{4}\right) = 8 + 8$
16 $\sec\left(\frac{7\pi}{4}\right) = \sqrt{2}$.

$T_3(x) = -1 + 2\left(x - \frac{7\pi}{4}\right) - \frac{4}{2!}\left(x - \frac{7\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{7\pi}{4}\right)^3$

$$\int_{5/4}^{6} \frac{4\,dx}{(4x-5)^{3/2}}$$ ← vertical asymptote at $x = 5/4$.



$$= \lim_{R\to \frac{5}{4}} \int_R^6 \frac{4}{(4x-5)^{3/2}}\,dx = \lim_{R\to \frac{5}{4}} \Big[-2\,(4x-5)^{-1/2}\Big]_R^6$$

$$4\,(4x-5)^{-3/2}$$

$$\lim_{R\to 5/4} -2\frac{1}{\sqrt{19}} + 2\,(4R-5)^{-1/2} = -\frac{2}{\sqrt{19}} + \lim_{R\to 5/4} \frac{2}{\sqrt{4R-5}} \to \infty \text{ as } R\to 5/4.$$

Q6 T_2 $x = 0.4$ e^x find $|e^x - T_2(x)|$ at $x = -0.2$

$f(x) = e^x$ $f(0.4) = e^{0.4}$

$f'(x) = e^x$ $f'(0.4) = e^{0.4}$

$f''(x) = e^x$ $f''(0.4) = e^{0.4}$

$$T_2 = e^{0.4} + e^{0.4}(x-0.4) + \frac{e^{0.4}}{2}(x-0.4)^2$$

$$\left| e^{-0.2} - \left(e^{0.4} + e^{0.4}(-0.6) + \frac{e^{0.4}}{2}(0.6)^2 \right) \right|.$$

$$\left| \cancel{f(0.2)} \; f(-0.2) - T_2(-0.2). \right|.$$

Q6 $\int_5^\infty \frac{1}{x^p \ln x}\,dx$ converge.

$$= \lim_{R\to\infty} \int_5^R \frac{1}{x^p \ln(x)}\,dx.$$

① $u = \ln(x)$. $e^u = x$.

$\frac{du}{dx} = \frac{1}{x}$. $\int \frac{1}{x^p u} \frac{dx}{du}\,du$.

$$= \int \frac{1}{x^p \cdot u} x\,du = \int \frac{x^{1-p}}{u}\,du.$$

$\int \frac{(e^u)^{1-p}}{u}\,du$.

② $\int \underbrace{x^{-p}} \underbrace{\frac{1}{\ln(x)}}\,dx$. $\int uv'\,dx = uv - \int u'v\,dx$.

observe. $(\ln(x))' = \frac{1}{x}$. $(\ln(\ln(x)))' = \frac{1}{\ln(x)} \cdot \frac{1}{x}$.

$$\int \frac{1}{x}$$

Know: $\int_1^\infty \frac{1}{x^p}\,dx \leftarrow$ converges for $p > 1$. diverges for $p \leq 1$.

$$= \lim_{R\to\infty} \int_1^R x^{-p}\,dx = \lim_{R\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^R = \lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

converges if $-p+1 < 0$ $p > 1$.

$\int_5^\infty \frac{1}{x^p \ln(x)}\,dx$ know $\int_5^\infty \frac{1}{x^p}\,dx$ converges if $p>1$, diverges if $p \le 1$.

$$\boxed{\frac{1}{x^p \ln(x)} < \frac{1}{x^p}}$$

$\int_5^\infty \frac{1}{x^p}\,dx$ converges for $p>1$

$\Rightarrow \int_5^\infty \frac{1}{x^p \ln(x)}$ converges for $p>1$.

$\boxed{(1,\infty)}$

answer.

$\underline{p=1}$ $\int_5^\infty \frac{1}{x\ln(x)}\,dx$ $\underline{\text{try}}$ $u=\ln(x)$

$\frac{du}{dx} = \frac{1}{x}$.

$$\lim_{R\to\infty} \int_{\ln(5)}^{\ln(R)} \frac{1}{x\cdot u}\frac{dx}{du}\,du = \lim_{R\to\infty}\int_{\ln(5)}^{\ln(R)} \frac{1}{x\cdot u}\, x\,du = \lim_{R\to\infty}\int_{\ln(5)}^{\ln(R)} \frac{1}{u}\,du$$

$$= \cancel{\lim_{R\to\infty} \int_{\ln(5)} \ln(u)\,dx} = \lim_{R\to\infty}\Big[\ln(u)\Big]_{\ln(5)}^{\ln(R)} = \lim_{R\to\infty} \ln(\ln(R)) - \ln(\ln(5))$$

$\longrightarrow \infty$ as $R \to \infty$

$$\boxed{\frac{1}{x\ln(x)} < \frac{1}{x^p \ln(x)}}$$

$p<1$.

$x^p < x$.

$\frac{1}{x^p} > \frac{1}{x}$.

$\Rightarrow$ diverges for $p<1$.

last time: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = 1$.

geometric series $c + cr + cr^2 + \dots = \frac{c}{1-r}$ as long as $|r| < 1$.

<u>special case</u> (Telescoping).

<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$

$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$

$s_1 = \frac{1}{2}$.

$s_2 = \frac{1}{2} + \frac{1}{6}$.

$s_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12}$.

$$\frac{1}{n(n+1)} = \frac{1}{n} + \frac{-1}{n+1}$$

$s_1 = 1 - \frac{1}{2}$

$s_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$.

$s_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$.

$s_N = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{N-1} - \frac{1}{N} + \frac{1}{N} - \frac{1}{N+1} = 1 - \frac{1}{N+1}$.

$$S_N = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{5} + \cdots - \frac{1}{N} + \frac{1}{N} - \frac{1}{N+1} = 1 - \frac{1}{N+1}.$$

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1.$$

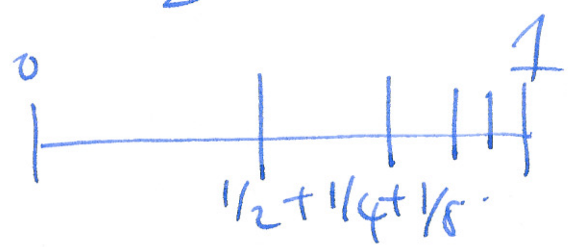
so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

<u>Example</u> (harmonic series) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}.$

$$\underbrace{1}_{\geq 1/2} + \underbrace{\frac{1}{2}}_{\geq 1/2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}} + \cdots$$

diverges!

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$



$$1 + 1 + 1 + 1 + 1 + \cdots \longrightarrow \infty.$$

Q: when does a series converge? A: don't know.

Tools

- Thm Divergence test If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

Warning $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

recall $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges but $\frac{1}{n} \to 0$ as $n \to \infty$.

- Rules for series / infinite sums

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series

then $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

$$\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

<u>observation</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^{2n}}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

§ 10.3 Positive series

positive series : $\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ all $a_n \geq 0$.

note : the partial sums S_N are an increasing sequence.

$S_{N+1} = S_N + \underbrace{a_{n+1}}_{\geq 0.}$ $\qquad S_{N+1} \geq S_N$.

recall : an increasing sequence with an upper bound converges.

Thm Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs:

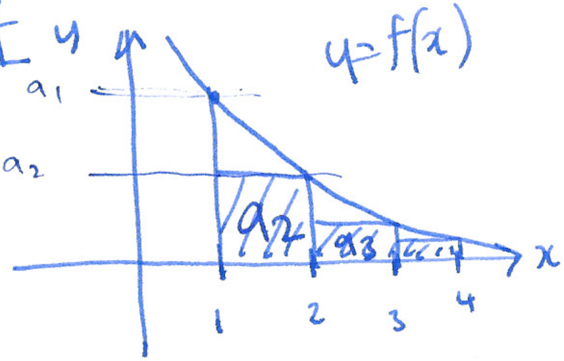
1. the partial sums S_N are bounded above, and S converges.
2. the partial sums are not bounded above, and S diverges.

Thm Integral test $\sum_{n=1}^{\infty} a_n$ where $a_n = f(n)$ $\left.\begin{array}{l} f(x) \text{ positive} \\ \text{decreasing} \\ \text{continuous} \end{array}\right\} x \geq 1$

then 1. if $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

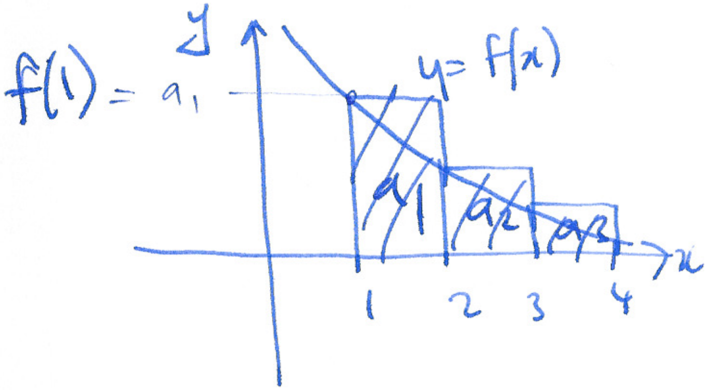
2. if $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof



$$S_N = a_1 + a_2 + \cdots + a_N.$$

$$S_N - a_1 = a_2 + a_3 + \cdots + a_N. \leq \int_1^N f(x)\,dx.$$



$$S_N = a_1 + a_2 + \cdots + a_N \geq \int_1^N f(x)\,dx$$

$\square$.

Example $S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ $\qquad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$.

$\sum_{n=1}^{\infty} a_n \qquad a_n = \frac{1}{\sqrt{n}}$. $\qquad$ ↑ converges $\Leftrightarrow \int_1^{\infty} x^{-1/2}\,dx$.

$$\lim_{R\to\infty} \int_1^R x^{-1/2}\,dx = \lim_{R\to\infty}\left[2x^{1/2}\right]_1^R = \lim_{R\to\infty} 2\sqrt{R} - 2 \to \infty$$

as $R \to \infty$.

diverges.

Thm convergence of p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

Proof (integral test) $\sum_{n=1}^{\infty} a_n$ $a_n = \frac{1}{n^p}$ $f(x) = \frac{1}{x^p} = x^{-p}$.

$\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if $p > 1$, diverges if $p \leq 1$ $\square$.

Examples

$p = 1$ $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

$p = 2$ $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$ converges. $\pi^2/6$.

$p = 3$ $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$ converges. ~~Rua~~ no-one knows to what?