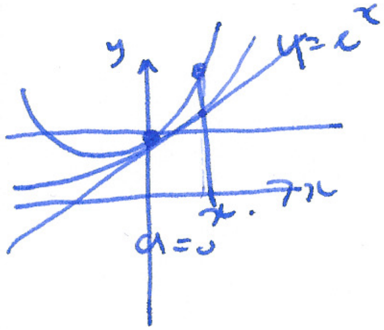
①

reminder: next class Tuesday (tomorrow).

last time: Taylor polynomials: e^x approx by a polynomial, $a=0$



deg 0 $\quad 1$

deg 1 $\quad 1+x$

deg 2 $\quad 1+x+\frac{x^2}{2}$

deg 3 $\quad 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$

$$T_n(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \dots$$
$$\dots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \dots$$

Q: how good are the Taylor approximations?

<u>Thm</u> (error bound) $f(x)$ function $f^{(n)}(x)$ exist and are cts.

let K be an upper bound for $|f^{(n+1)}(u)|$ for all u in $[a,x]$.

then $|T_n(x) - f(x)| \leq K\frac{|x-a|^{n+1}}{(n+1)!}$

<u>Example</u> $f(x) = \sin(x)$ $\quad \underset{\text{centered at } 0.}{T_3(x)} = x - \frac{x^3}{3!}$ $\qquad$ centered at $a=0$: $T_3(1) = 1 - \frac{1}{3!} = 1 - \frac{1}{6} = \frac{5}{6}$.

$\left.\begin{array}{l}|\sin(x)| \leq 1 \\ |\cos(x)| \leq 1\end{array}\right\}$ can choose $K=1$ $\qquad |T_3(1) - f(1)| \leq \frac{1\,|1-0|^4}{4!} = \frac{1}{4!} = \frac{1}{24}$.

§10.1 Sequences

a $\cancel{f}: \mathbb{N} \to \mathbb{R}$.

Defn A sequence is a list of numbers indexed by $\mathbb{N}$ ← natural numbers $\{1, 2, 3, \dots\}$.

Examples $1, 2, 3, 4, \dots$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ $\quad 1, 1, 1, 1, \dots$ $\quad 1, 27, \pi, \frac{1}{e}, \dots$

notation : $a_1, a_2, a_3, \dots$ or (a_n) $\quad (a_n)_{n \in \mathbb{N}}$

Q: what is not a sequence? a single number, a set of numbers, function $f: \mathbb{R} \to \mathbb{R}$.

set $\{2, 4, 8\}$

sometimes (but not always) can give a sequence by a formula

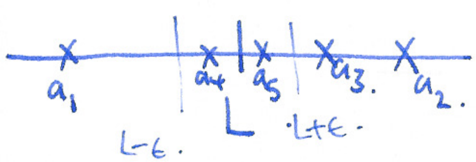
$(a_n)_{n \in \mathbb{N}}$ $\quad a_n = n$ $\quad (n)_{n \in \mathbb{N}}$ $\quad 1, 2, 3, \dots$

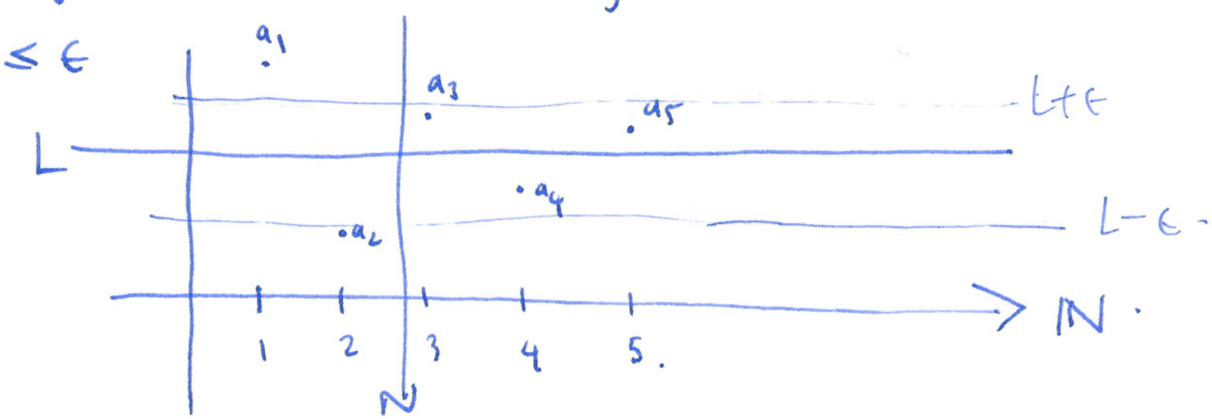
$(a_n)_{n \in \mathbb{N}}$ $\quad a_n = \frac{1}{1+n^2}$ $\quad \frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \dots$

Example (recursively defined sequence) Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$

$a_1 = 1, \ a_2 = 1, \quad a_{n+2} = a_n + a_{n+1}$

Defn A sequence $(a_n)_{n \in \mathbb{N}}$ converges to L if for every $\epsilon > 0$, there is a N s.t. for all $n \geq N$ $\quad |a_n - L| \leq \epsilon$





<u>Example</u> $a_n = \frac{1}{n}$ $\quad 1, \frac{1}{2}, \frac{1}{3}, \dots$ converges to 0

<u>Proof</u> given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$. $\Rightarrow \frac{1}{N} < \epsilon$.

if $n \geq N$, then $\underbrace{\frac{1}{n}}_{a_n} \leq \frac{1}{N} < \epsilon$. $\square$.

aside

$f : \mathbb{R} \to \mathbb{R}$ ← real numbers.

$f : \mathbb{N} \to \mathbb{R}$

↑ natural numbers.

$\mathbb{N}$ $\quad 1, 2, 3, 4, \dots$

$f(1), f(2), f(3), f(4), \dots$

<u>special case</u> : sequence defined by a function.

$f(x)$ $\quad f : \mathbb{R} \to \mathbb{R}$ $\qquad a_n = f(n)$

<u>Thm</u> If $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} f(n) = L$.

Q: is the converse true? A: no. $\quad f(x) = \sin(\pi x)$

$\lim_{n \to \infty} \sin(\pi x)$ DNE.

$a_n = \sin(n\pi) = 0$. $\lim_{n \to \infty} a_n = 0$.

<u>Example</u> $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

$a_n = \frac{n-1}{n} \leftarrow f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$.

$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n \to \infty} \frac{n-1}{n} = 1$.

Examples (geometric ~~serie~~ sequences) $a_n = r^n$.

e.g. 2^n: $2, 4, 8, 16, \ldots$

$\left(\frac{1}{3}\right)^n$: $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

1^n: $1, 1, 1, 1, \ldots$

$(-1)^n$: $-1, 1, -1, 1, -1, 1, \ldots$

$(-2)^n$: $-2, 4, -8, 16, \ldots$

Fact $\lim_{n \to \infty} r^n$

	limit
$r > 1$	∞
$r = 1$	1
$\lvert r\rvert < 1$	0
$r \leq -1$	DNE

rules for limits of sequences: same as rules for limits of functions

suppose $a_n \to L$ and $b_n \to M$. $\leftarrow$ notation: means $\lim_{n \to \infty} b_n = M$.

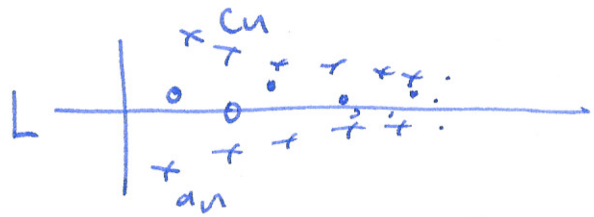
- $\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n = L + M$
- $\lim_{n \to \infty} (a_n b_n) = \left(\lim_{n \to \infty} a_n\right)\left(\lim_{n \to \infty} b_n\right) = LM$
- $\lim_{n \to \infty} \left(a_n / b_n\right) = \left(\lim_{n \to \infty} a_n\right) / \left(\lim_{n \to \infty} b_n\right) = L/M \leftarrow M \neq 0$.
- $\lim_{n \to \infty} c\, a_n = c \lim_{n \to \infty} a_n = cL$ $\quad$ c constant $\leftarrow$ doesn't depend on n.

<u>Example</u> $a_n = \frac{1}{n}$ $\quad b_n = \frac{n-1}{n}$ $\quad a_n \to 0 \quad b_n \to 1$.

$$\lim_{n\to\infty} a_n+b_n = \lim_{n\to 0} \frac{1}{n} + \frac{n-1}{n} = \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} \frac{n-1}{n} = 0+1 = 1.$$

<u>Squeeze Thm</u> If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = L$ $\quad \lim_{n\to\infty} b_n = L$

then $\lim_{n\to\infty} b_n = L$ $\square$



<u>Example</u> $a_n = \frac{R^n}{n!}$ $\quad (R \in \mathbb{R})$ real numbers. $\mathbb{R}$.

<u>Fact</u> $\lim_{n\to\infty} \frac{R^n}{n!} = 0$.

<u>Proof</u> there is an integer M s.t. $\quad M \le R \le M+1$.

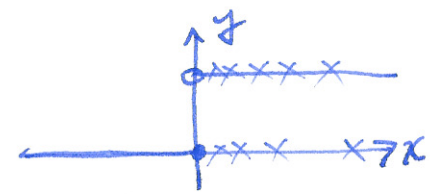
$$0 \le \frac{R^n}{n!} = \underbrace{\frac{R}{1}\cdot\frac{R}{2}\cdots\cdots\frac{R}{M}}_{\text{call this } A} \underbrace{\frac{R}{M+1}\cdots\cdots\frac{R}{n-1}}_{\le 1}\cdot\frac{R}{n} \le A\frac{R}{n}.$$

$0 \le \frac{R^n}{n!} \le A\frac{R}{n}$. $\quad \lim_{n\to\infty} 0 = 0 \quad \lim_{n\to\infty} \frac{A \cdot R}{n} = 0$.

$\Rightarrow \lim_{n\to\infty} \frac{R^n}{n!} = 0$ (by the squeeze thm).

<u>Thm</u> If $f(x)$ is continuous at $x = L$ and $\lim_{n\to\infty} a_n = L$

then $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$.



important $f(x)$ cts at L. Bad example: $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 1 \end{cases}$

$a_n = \frac{1}{n} \quad a_n \to 0 \quad f(a_n) = 1 \quad \lim_{n\to\infty} f(a_n) = 1$. ✗

$f\left(\lim_{n\to\infty} a_n\right) = f(0) = 0$.

<u>Example</u> find $\lim_{n\to\infty} e^{n/(n+1)}$

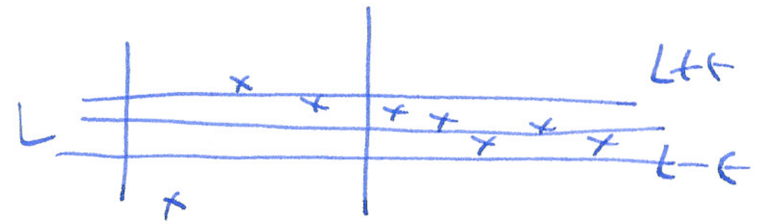
$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1 \quad \lim_{n\to\infty} e^{n/(n+1)} = e^{\lim_{n\to\infty} n/(n+1)} = e^{1} = e$

Def^n A sequence (a_n) is **bounded above** if $a_n \leq M$ for all n

bounded below if $L \leq a_n$ for all n.

bounded if $L \leq a_n \leq M$ for all n.

Thm Convergent sequences are bounded.

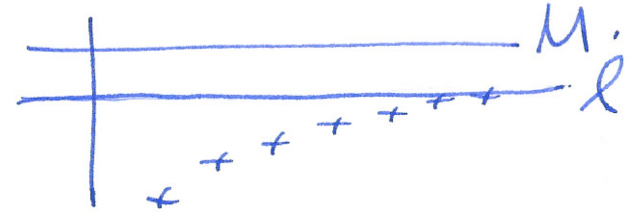


warning: bounded sequences need not converge.

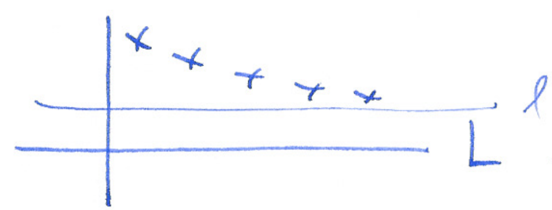
Example $0, 1, 0, 1, 0, 1, 0, 1, \ldots \leftarrow a_n = \frac{1-(-1)^n}{2}$.

Thm Bounded monotonic sequences converge.

↑ either increasing or decreasing



- if (a_n) is increasing and $a_n \leq M$ then $a_n \to \ell \leq M$.
- if (a_n) is decreasing and $L \leq a_n$ then $a_n \to \ell \geq L$



Example $a_n = \frac{1}{n}$. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ claim: decreasing $a_{n+1} \leq a_n$. (*)

$a_n \geq a_{n+1}$ ✓.

$n < n+1$ $\quad \frac{1}{n} > \frac{1}{n+1}$ $\quad a_n > a_{n+1}$ ← so (a_n) decreasing

so $L = -1$ is a lower bound so $\lim_{n\to\infty} a_n = l \geq -1$.

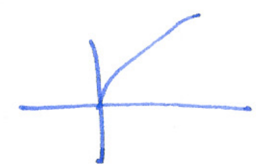
(*) decreasing means $a_n \geq a_{n+1}$ for all n..

strictly decreasing $a_n > a_{n+1}$ for all n.

Example $a_n = \sqrt{n+1} - \sqrt{n}$ show . decreasing, bounded below.

bounded below ~~decreasing~~ :

$n+1 > n$ $\quad \sqrt{x}$ is monotonic, increasing



$\sqrt{n+1} > \sqrt{n}$

$\sqrt{n+1} - \sqrt{n} > 0$ so can choose $L=0$ as lower bound.

decreasing: consider $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - x^{1/2}$

$$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right).$$

claim: $f'(x) < 0$ where $f'(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$.

$x+1 > x$

$\sqrt{x+1} > \sqrt{x}$.

$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$. $\Rightarrow$ $\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}} < 0$ $\Rightarrow$ $f'(x) < 0$ decreasing.

so $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$ exists $= l \geq 0$.

§ 10.2 Series

Defn A series is an infinite sum $a_1 + a_2 + a_3 + a_4 + \cdots = \sum_{n=1}^{\infty} a_n$.

Examples $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ $\quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ $\quad 1+1+1+1+1+\cdots$ $\quad 1-1+1-1+1-1\cdots$

Defn The N-th partial sum $S_N = a_1 + a_2 + \cdots + a_N$

Defn The sum of the infinite series is defined to be the limit of the sequence of partial sums, if that limit exists.

so $$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

if $\lim_{N\to\infty} S_N = S$ then we say $\sum_{n=1}^{\infty} a_n$ <u>converges</u> and $\sum_{n=1}^{\infty} a_n = S$.

<u>Examples</u> ① $1+1+1+1+\cdots$ $S_N = \underbrace{1+1+\cdots+1}_{N} = N$ $\lim_{N\to\infty} N = \infty$

sum does not converge.

② $1-1+1-1+1-1+\cdots$

↑ does not converge.

$s_1 = 1$
$s_2 = 0$
$s_3 = 1$
$s_4 = 0$
$\vdots$

$(s_N) = 1, 0, 1, 0, 1, 0, 1, \ldots$

(s_N) does not converge.

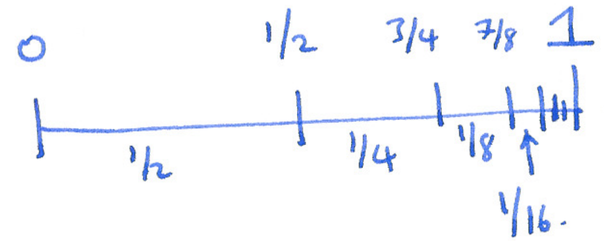
<u>warning</u> can't necessarily re-arrange terms

$$\underbrace{(1-1)}_{0}+\underbrace{(1-1)}_{0}+\underbrace{(1-1)}_{0}+\cdots = 0.$$

$$1+\underbrace{(-1+1)}_{0}+\underbrace{(-1+1)}_{0}+\underbrace{(-1+1)}_{0}+\cdots = 1.$$

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty}\frac{1}{2^n}$$



$$s_1 = \frac{1}{2} = 1-\frac{1}{2}$$

want $S_N = 1-\frac{1}{2^N}$.

$$s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1-\frac{1}{4}.$$

$$s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1-\frac{1}{8}.$$

$$s_4 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = \frac{15}{16}.$$

$$\frac{1}{2}\times\frac{1}{2^N} = \frac{1}{2^{N+1}}$$

$$S_N = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^{N-1}}+\frac{1}{2^N}.$$

$$\frac{1}{2}S_N = \frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^N}+\frac{1}{2^{N+1}}$$

$$S_N-\frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}} \longrightarrow \frac{1}{2}S_N = \frac{1}{2}-\frac{1}{2^{N+1}}$$

$$S_N = 1-\frac{1}{2^N} \to \lim_{N\to\infty} 1-\frac{1}{2^N} = \lim_{N\to\infty} 1 + \lim_{N\to\infty} -\frac{1}{2^N}$$

$$1 + 0 = 1.$$

general case

$$c + cr + cr^2 + cr^3 + \cdots = \sum_{n=0}^{\infty} cr^n$$

$$S_N = c + cr + cr^2 + \cdots + cr^N$$

$$rS_N = cr + cr^2 + cr^3 + \cdots + cr^N + cr^{N+1}$$

$$S_N - rS_N = c - cr^{N+1}.$$

$$S_N(1-r) = c(1-r^{N+1}).$$

$$S_N = \frac{c(1-r^{N+1})}{1-r}$$

$$\lim_{N\to\infty} \frac{c(1-r^{N+1})}{1-r} = \frac{c}{1-r}$$

as long as $|r| < 1$.

diverges if $|r| \geq 1$.