7.5Q6

$$\int \frac{x^2-25}{x^2+25}\,dx \qquad \begin{array}{r|l} & 1 \\ \hline x^2+25 & x^2-25 \\ & x^2+25 \\ \hline & -50 \end{array} \qquad \int 1 - \frac{50}{x^2+25}\,dx$$

know. $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$

(1)

$$\underbrace{\int 1\,dx}_{x} - \int \frac{50}{x^2+25}\,dx$$

~~u=~~ $x = 5u$

$\frac{dx}{du} = 5.$

$$x - \int \frac{50}{25u+25}\frac{dx}{du}\,du =$$

$$x - \int \frac{50}{25u^2+25}\,5\,du = x - \int \frac{2}{u^2+1}\cdot 5\,du = x - 10\int \frac{1}{1+u^2}\,du$$

$$= x - 10\tan^{-1}(u)+c = x - 10\tan^{-1}(x/5)+c$$

7.5Q1.

$$\frac{-5x^2-15x-24}{(x+4)(x^2+6)} = \frac{A}{x+4} + \frac{Bx+C}{x^2+6} = \frac{A(x^2+6)+(Bx+C)(x+4)}{(x+4)(x^2+6)}.$$

$$= \frac{x^2(A+B)+x(4B+C)+6A+4C}{(x+4)(x^2+6)}$$

$-5 = A+B$ (1)
$-15 = 4B+C$ (2)
$-24 = 6A+4C$ (3)

(3) − 4(2).

$-24+60 = 6A - 16B \cancel{+4C} \cancel{-4C}$

(2) $-15 = -12 + C$

$C = -3.$

$36 = 6A - 16B.$
$-80 = 16A + 16B.$

$-44 = 22A$

$A = -2$

(1) $-5 = -2 + B$ $\quad B = -3.$

$$\int \frac{10}{(x+1)(x^2+9)^2}\,dx = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} + \frac{Dx+E}{(x^2+9)^2} = \frac{A(x^2+9)^2 + (Bx+C)(x+1)(x^2+9) + (Dx+E)(x+1)}{(x+1)(x^2+9)^2}.$$

consider $x=-1$ : $10 = A100$ $\quad A = \frac{1}{10}$.

$$10 = \frac{1}{10}(x^2+9)^2 + (Bx+C)(x+1)(x^2+9) + (Dx+E)(x+1)$$

$$10 = \frac{1}{10}(x^4+18x^2+81) + (Bx+C)(x^3+x^2+9x+9) + Dx^2 + (D+E)x + E.$$

$$10 = x^4\left(\frac{1}{10}+B\right) + x^3(C+B) + x^2\left(\frac{9}{5} + 9B + C + D\right) + x(9B+9C+D+E) + \frac{81}{10} + 9C + E.$$

$\frac{1}{10}+B=0 \Rightarrow B = -1/10.$

$C+B=0 \Rightarrow C = 1/10.$

$\frac{9}{5}+9B+C+D=0 \Rightarrow \frac{18}{10}\cancel{\frac{9}{5}} - 9/10 + 1/10 + D = 0 \quad D=-1.$

$9B+9C+D+E=0$

$\frac{81}{10}+9C+E = 10. \quad \frac{81}{10} + \frac{9}{10} + E = 10 \quad \frac{90\,\cancel{89}}{10} + E = 10. \quad E = \frac{100-90\,\cancel{89}}{10} = \frac{10}{10}.$

$E=1.$

$$\int \frac{10}{(x+1)(x^2+9)^2}\,dx = \int \frac{1/10}{x+1} + \frac{-1/10\,x + 1/10}{x^2+9} + \frac{-x+1}{(x^2+9)^2}\,dx.$$

$$\circledast \int \frac{1/10}{x+1} + \frac{-1/10\,x + 1/10}{x^2+9} + \frac{-x+1}{(x^2+9)^2}\,dx = \frac{1}{10}\ln|x+1| + \cdots$$

$$\int \frac{-1/10\,x}{x^2+9} + \frac{1/10}{x^2+9}\,dx = -\frac{1}{20}\ln|x^2+9| + \boxed{\frac{1}{10}\int \frac{1}{x^2+9}\,dx}$$

~~att~~ $x = 3u$. $\frac{dx}{du} = 3$.

$$\boxed{\frac{1}{10}\int \frac{1}{9u^2+9}\frac{dx}{du}\,du = \frac{1}{90}\int \frac{1}{u^2+1}\,3\,du = \frac{1}{30}\int \frac{1}{1+u^2}\,du = \frac{1}{30}\tan^{-1}(u) + c = \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right)}$$

$$\circledast = \frac{1}{10}\ln|x+1| - \frac{1}{20}\ln|x^2+9| + \frac{1}{30}\tan^{-1}\left(\frac{x}{3}\right) + \underbrace{\int \frac{-x+1}{(x^2+9)^2}\,dx}.$$

$$\int \frac{-x}{(x^2+9)^2}\,dx + \int \frac{1}{(x^2+9)^2}\,dx = \cancel{-\frac{1}{2}\ln|}\ -\frac{1}{2}(x^2+9)^{-1} + \int \frac{1}{(x^2+9)^2}\,dx.$$

$\boxed{7.5.\ Q7 \leftarrow \text{cancel.}}$

$$\int \frac{1}{(1+x^2)^2}\,dx \leftarrow \quad x = \tan u. \quad \frac{dx}{du} = \sec^2 u.$$

$\sin^2 u + \cos^2 u = 1$

$\tan^2 u + 1 = \sec^2 u.$

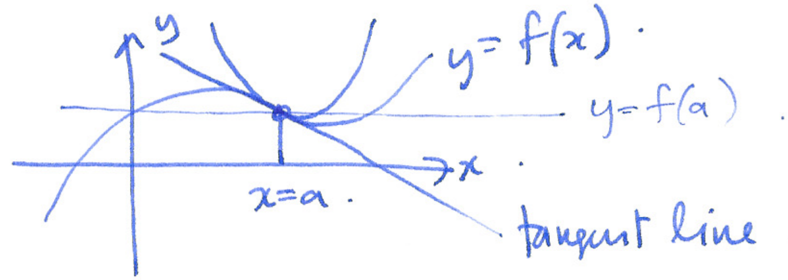
$$\int \frac{1}{(1+\tan^2 u)^2}\frac{dx}{du}\,du. \rightsquigarrow \int \frac{1}{\sec^4 u}\sec^2 u\,du = \int \frac{1}{\sec^2 u}\,du = \int \cos^2 u\,du.$$

$\uparrow$ $\sec^2 u.$

!!!! write up a sol$^{n}$ and email you.

# §8.4 Taylor polynomials

Approximating functions close to $x=a$




0-th approx: $f(x) \approx f(a)$

1-st approx: straight line / tangent line. $f(x) \approx f(a) + f'(a)(x-a)$.

2nd approx: ? quadratic $\}$ how do we find them?

3rd approx: ? cubic

tangent line: unique line with same value as $f$ at $x=a$
same slope as $f$ at $x=a$.

quadratic approx: unique quadratic with same ~~slope~~ value as $f$ at $x=a$
same slope as $f$ at $x=a$
same 2nd derivative as $f$ at $x=a$.

cubic approx: same first three derivatives at $x=a$.

quadratics

$T_2(x) = ax^2 + bx + c$

better: $T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$.

$T_2(x) = \alpha x^2 + bx + c$

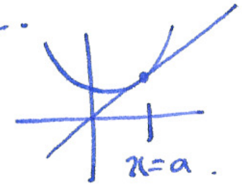
$T_2(a) = \alpha a^2 + ba + c$.

$T_2'(x) = 2\alpha x + b$.

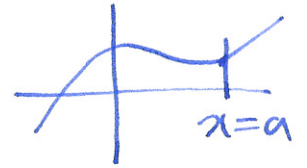
$T_2'(a) = 2\alpha a + b$.

$T_2''(x) = 2\alpha$.

$T_2''(a) = 2\alpha$.

$$T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2.$$ 



$f(x)$. 



value at $x=a$: $T_2(a) = a_0$. $\qquad a_0 = f(a)$.

slope $x=a$: $T_2'(x) = a_1 + 2a_2(x-a)$.

$T_2'(a) = a_1$. $\qquad a_1 = f'(a)$.

2nd derivative $x=a$: $T_2''(x) = 2a_2$ $\qquad 2a_2 = f''(a)$.

$T_2''(a) = 2a_2$. $\qquad \Leftrightarrow a_2 = \frac{1}{2} f''(a)$.

so $T_2(x) = \underbrace{f(a)}_{a_0} + \underbrace{f'(a)}_{a_1}(x-a) + \underbrace{\frac{1}{2} f''(a)}_{a_2}(x-a)^2.$ $\leftarrow$ 2nd order Taylor polynomial at $x=a$.

$\uparrow$ quadratic w/ same value as $f(x)$ at $x=a$

slope " "

2nd derivative " " .

In general $\quad T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$

differentiate $k$ times: $\quad T_n^{(k)}(x) = k!\,a_k +$ other terms all which contain $(x-a)$.

$x^k$

$k x^{k-1}$ $\qquad T_n^{(k)}(a) = k!\,a_k = f^{(k)}(a)$ ~~$a_k =$~~

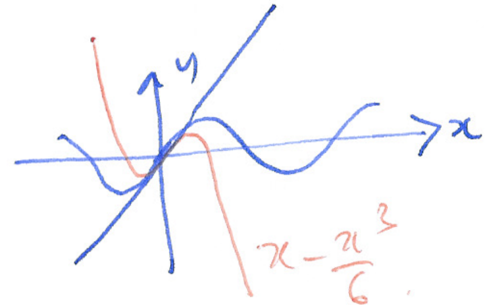
$k(k-1)x^{k-2}$

$\vdots$

$k!\,x^0$ $\qquad T_n(x) = a_0 + a_1(x-a) + \dots + a_n(x-a)^n \qquad a_k = \dfrac{f^{(k)}(a)}{k!}$

so $T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$




**Example** ① $y = \sin x$, at $a = 0$

$f(x) = \sin(x) \qquad f(0) = 0$

$f'(x) = \cos(x) \qquad f'(0) = 1$

$f''(x) = -\sin(x) \qquad f''(0) = 0$

$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$

$$T_4(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!}$$

$$= 0 + 1.x + 0.\frac{x^2}{2!} - 1.\frac{x^3}{3!} + 0.\frac{x^4}{4!}$$

$$= x - \frac{x^3}{6}$$

② $y = \cos(x)$, at $a = 0$

$f(x) = \cos(x) \qquad f(0) = 1$

$f'(x) = -\sin(x) \qquad f'(0) = 0$

$f''(x) = -\cos(x) \qquad f''(0) = -1$

$f^{(3)}(x) = \sin(x) \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$

$$T_4(x) = \underbrace{\overset{f(0)}{1}}_{a_0} + \underbrace{0}_{a_1}.x + \underbrace{(-1)}_{a_2}\frac{x^2}{2!} + \underbrace{0}_{a_3}.\frac{x^3}{3!} + \underbrace{1}_{a_4}.\frac{x^4}{4!}$$

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} .$$

$a_0 = f(0)$

$a_1 = f'(0)$

$a_2 = f''(0)/2!$

$a_3 = f^{(3)}(0)/3!$

(3) $f(x) = e^x$  $a=0$.

$i^2 = -1$.

$f(x) = e^x$  $f(0) = 1$  $T_2(x) = 1 + x + \frac{1}{2!}x^2$

$f'(x) = e^x$  $f'(0) = 1$  $T_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$.

$f''(x) = e^x$  $f''(0) = 1$  $T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$i\theta$

$(i\theta)^2 = i^2\theta^2 = -\theta^2$

$(i\theta)^3 = i^3\theta^3 = -i\theta^3$.

$(i\theta)^4 = i^4\theta^4 = \theta^4$.

sneak preview  what if we continue this pattern for ever?

$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$  ← not a polynomial.

Fact.

$$\left.\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}\right\} \text{radians!}$$

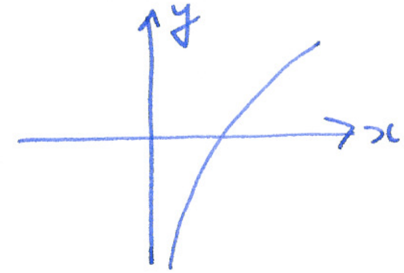
Remark

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \quad \cos\theta$$

$$+ i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} \quad + i\sin\theta.$$

$e^{i\theta} = \cos\theta + i\sin\theta$  $e^{i\pi} = -1$.




**Example** $y = \ln(x)$ $\quad a = 1$.

$$f(x) = \ln(x) \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \qquad f'(1) = 1$$

$$f''(x) = -x^{-2} \qquad f''(1) = -1$$

$$f^{(3)}(x) = 2x^{-3} \qquad f^{(3)}(1) = 2$$

$$f^{(4)}(x) = -6x^{-4} \qquad f^{(4)}(1) = -3!$$

$$f^{(k)}(x) = \cancel{(-1)^{k+1}(k-1)!\,x^{-k}}\ (-1)^{k+1}(k-1)!\,x^{-k}. \qquad f^{(k)}(1) = (-1)^{k+1}(k-1)!$$

$$T_3 = 0 + 1.(x-1) - 1\frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!}.$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.$$

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n.$$