MTH 232 last time: partial fractions.

§7.6 Strategies for integration.

tools: • rewrite expressions. e.g. $\dfrac{x^3-1}{x-1}$ ← long division. $= \dfrac{(x-1)(x^2+x+1)}{x-1}$

$= x^2+x+1$ ← integrate that.

e.g. $\dfrac{x-x^3}{\sqrt{x}} = x^{1/2} - x^{5/2}$ ← integrate.

- trig identities
- partial fractions.
- substitutions
- integration by parts.

Example 1

$$\int x^3\sqrt{1+x^2}\,dx.$$ try $u = 1+x^2 \leftarrow x^2 = u-1$

$$\frac{du}{dx} = 2x.$$

$$\int x^3\sqrt{u}\,\frac{dx}{du}\,du = \int x^{\cancel{3}^2} u^{1/2}\cdot\frac{1}{2\cancel{x}}\,du = \frac{1}{2}\int x^2 u^{1/2}\,du.$$

$$= \frac{1}{2}\int (u-1)u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du = \frac{2}{10}u^{5/2} - \frac{2}{6}u^{3/2} + c.$$

2 $$\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx.$$ try $u = \sqrt{x}+1$ $\leftarrow \sqrt{x} = u-1$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$$

$$\int \frac{1}{\sqrt{u}}\,\frac{dx}{du}\,du = \int \frac{1}{\sqrt{u}}\cdot 2\sqrt{x}\,du$$

$$\int \frac{1}{\sqrt{u}}\,2(u-1)\,du = 2\int u^{1/2} - u^{-1/2}\,du = \frac{4}{3}u^{3/2} - 4u^{1/2} + c$$

know. $\int \sqrt{u^2+1}\,du.$

3 $$\int \sqrt{x^2+2x+2}\,dx$$ 1 complete the square: $$\int \sqrt{(x+1)^2+1}\,dx.$$

now do trig sub $x+1 = \tan\theta \longrightarrow$.

2 do this trig sub

3 integrate resulting trig functions...

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\sqrt{1+x^2}$ 2 $\qquad \sqrt{1-x^2}$ 1 $\qquad \sqrt{x^2-1}$ 3

$$\sin^2\theta + \cos^2\theta = 1.$$
$$\cos^2\theta = 1 - \sin^2\theta \quad 1.$$
$$\tan^2\theta + 1 = \sec^2\theta. \quad 2.$$
$$\tan^2\theta = \sec^2\theta - 1 \quad 3.$$

$$\int \sqrt{(x+1)^2+1}\, dx \qquad x+1 = \tan u.$$
$$\frac{dx}{du} = \sec^2 u.$$

$$\int \sqrt{\underbrace{\tan^2 u + 1}_{\sec^2 u}} \cdot \frac{dx}{du}\, du = \int \sec u . \sec^2 u \, du = \int \sec^3 u \, du$$

↑ parts.

$$\int \underbrace{\sec u}_{f} \underbrace{\sec^2 u}_{g'} du. \qquad \int f g' dx = fg - \int f' g \, dx.$$

$f = \sec u \qquad g' = \sec^2 u$

$f' = \sec u \tan u \qquad g = \tan u$

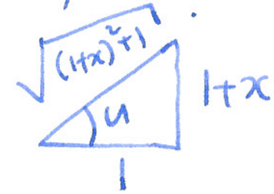
$$\int \sec^3 u \, du = \sec u \tan u - \int \sec u \underbrace{\tan^2 u}_{\sec^2 u - 1} du$$

$$\int \sec^3 u \, du = \sec u \tan u - \int \sec^3 u - \sec u \, du.$$

$$2\int \sec^3 u\,du = \sec u\tan u + \int \sec u\,du = \sec u\tan u + \ln|\sec u+\tan u| + C$$

$$\int \sec^3 u\,du = \frac{1}{2}\underbrace{\sec u}_{\sqrt{(1+x)^2+1}}\,\underbrace{\tan u}_{1+x} + \frac{1}{2}\ln\Big|\sec u + \underbrace{\tan u}_{1+x}\Big| + C$$

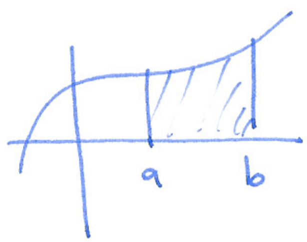
$\tan u = 1+x$.



$$= \frac{1}{2}\sqrt{(1+x)^2+1}\,(1+x) + \frac{1}{2}\ln\left|\sqrt{(1+x)^2+1} + 1+x\right| + C.$$

§7.7 Improper integrals

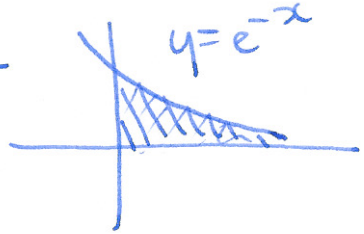
recall $\int_a^b f(x)\,dx$



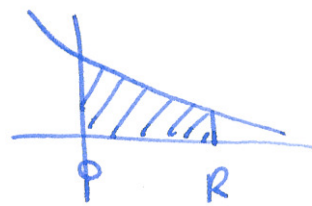
= area under the curve

Q: what about infinite intervals?

Example



$\int_0^\infty e^{-x}\,dx$

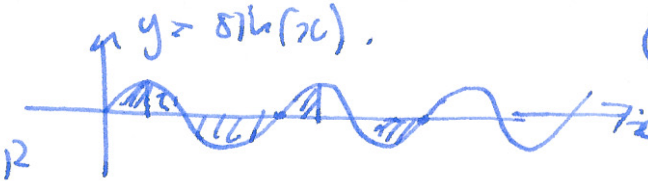


$$\lim_{R\to\infty}\int_0^R e^{-x}\,dx = \left[-e^{-x}\right]_0^R = -e^{-R}+1.$$

Def^{n} $\displaystyle\int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$. if this limit exists

otherwise say DNE.

$f(x)$ cgs.

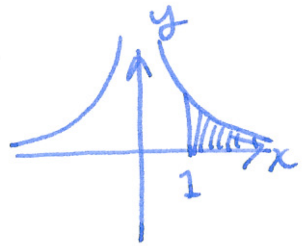


warning : sometime limit doesn't exist

$$\int_0^\infty \sin(x)\,dx = \lim_{R\to\infty}\int_0^R \sin(x)\,dx = \lim_{R\to\infty}\left[-\cos(x)\right]_0^R = \lim_{R\to\infty} -\cos(R)+1$$

DNE.

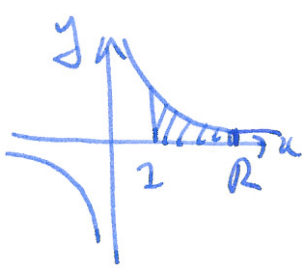
Example ① $\int_1^\infty \frac{1}{x^2}\,dx$



$$\lim_{R\to\infty}\int_1^R x^{-2}\,dx = \lim_{R\to\infty}\left[-x^{-1}\right]_1^R$$

$$= \lim_{R\to\infty} -\frac{1}{R}+1 = 1$$

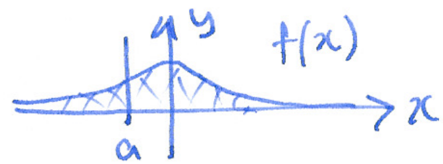
② $\int_1^\infty \frac{1}{x}\,dx$

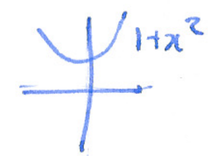


$$\lim_{R\to\infty}\int_1^R x^{-1}\,dx = \lim_{R\to\infty}\left[\ln|x|\right]_1^R = \lim_{R\to\infty} \ln(R) - 0$$

$= \infty$ (limit DNE)

Doubly infinite integrals $\int_{-\infty}^\infty f(x)\,dx$ $f(x)$, cts.





Defn (f cts) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$ provided both of these exist.

Example. $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$

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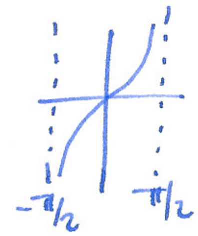
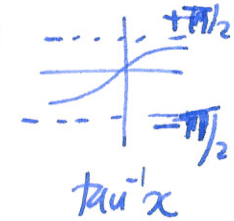
$$= \lim_{R\to\infty} \int_{-R}^{0} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_{0}^{R} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \Big[\tan^{-1}(x) \Big]_{-R}^{0} + \lim_{R\to\infty} \Big[\tan^{-1}(x) \Big]_{0}^{R}.$$

$$= \lim_{R\to\infty} \Big(0 - \tan^{-1}(-R) \Big) + \lim_{R\to\infty} \Big(\tan^{-1}(R) - 0 \Big) =$$

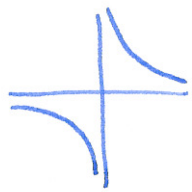




$$= \lim_{R\to\infty} \underbrace{\tan^{-1}(R)}_{\pi/2} + \lim_{R\to\infty} \underbrace{\tan^{-1}(R)}_{\pi/2} = \pi.$$

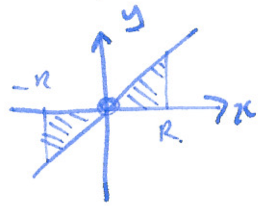
__warning__ : $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$ WRONG!

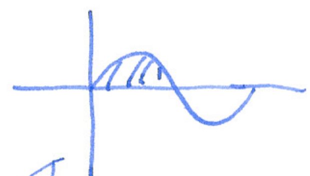
__Example__ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x}\,dx$

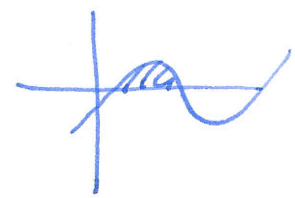


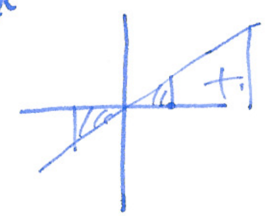
not cts! ← more complicated.

$\displaystyle\int_{-\infty}^{\infty} x\,dx$









$$\neq \lim_{R\to\infty} \int_{-R}^{R} x\,dx$$

$$= \boxed{\lim_{R\to\infty} \int_{-R}^{0} x\,dx} + \lim_{R\to\infty} \int_{0}^{R} x\,dx \quad \text{DNE}.$$

DNE DNE

$$\lim_{R\to\infty} \Big[\tfrac{1}{2}x^2 \Big]_{-R}^{0}$$

$$\lim_{R\to\infty} -\tfrac{1}{2}R^2 \to -\infty.$$

DNE.

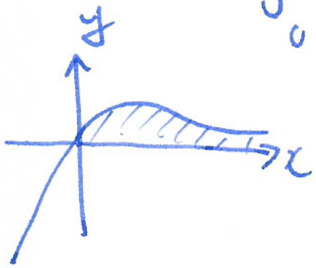
Example $\displaystyle\int_0^\infty x e^{-x}\,dx = \lim_{R\to\infty}\left(\int_0^R \underbrace{x}_{u}\,\underbrace{e^{-x}}_{v'}\,dx\right).$

$\boxed{\displaystyle\int uv'\,dx = uv - \int u'v\,dx.}$ ⑦

$u = x \qquad v' = e^{-x}$

$u' = 1 \qquad v = -e^{-x}$

recall.



$$= \lim_{R\to\infty}\left(\left[-x e^{-x}\right]_0^R - \int_0^R -e^{-x}\,dx\right).$$

$$\lim_{R\to\infty} -Re^{-R} + 0 + \left[-e^{-x}\right]_0^R.$$

$$\lim_{R\to\infty} \underbrace{-Re^{-R}}_{\to 0} \underbrace{- e^{-R}}_{\to 0} + 1 \;. = 1.$$

$(uv)' = u'v + uv'$

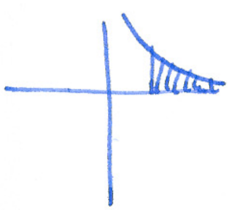
$\displaystyle\int (uv)'\,dx = \int u'v + uv'\,dx$

$\displaystyle uv = \int u'v\,dx + \int uv'\,dx$

$\displaystyle\int uv'\,dx = uv - \int u'v\,dx.$

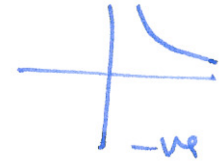
Remark. $\displaystyle\int_{-\infty}^0 f(x)\,dx = \lim_{R\to-\infty} \int_R^0 f(x)\,dx = \lim_{R\to+\infty} \int_{-R}^0 f(x)\,dx.$

Example when does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ exist? $\quad p=1$ no. $\quad p=2$ yes.

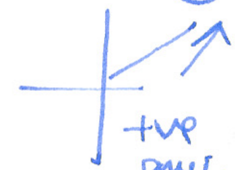


$$\lim_{R\to\infty}\int_1^R x^{-p}\,dx = \lim_{R\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^R = \lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}$$

$$\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{1-p} \qquad \lim_{R\to\infty} R^{-p+1} = 0 \text{ if } p>1$$
$$= \infty \text{ if } p<1.$$



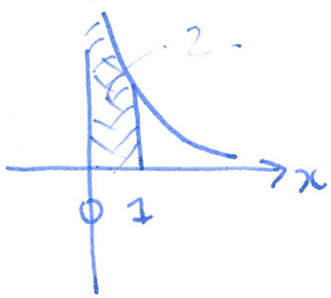
-ve pow



+ve power.

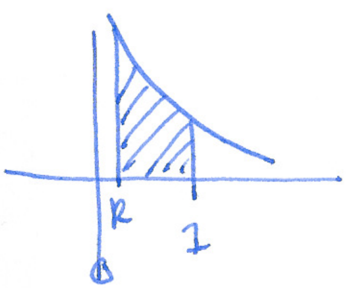
conclusion: $\int_1^\infty \frac{1}{x^p}\,dx$ exists if $p>1$
DNE if $p \le 1$.

Integrals with discontinuities



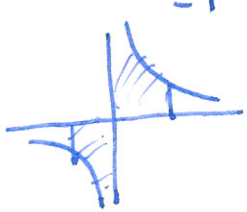
$y = \frac{1}{\sqrt{x}}$. $\int_0^1 \frac{1}{\sqrt{x}}\,dx \leftarrow$ this is improper! $f(0)$ not defined.

$$\lim_{R\to 0} \int_R^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0} \int_R^1 x^{-1/2}\,dx = \lim_{R\to 0} \left[2x^{1/2}\right]_R^1.$$



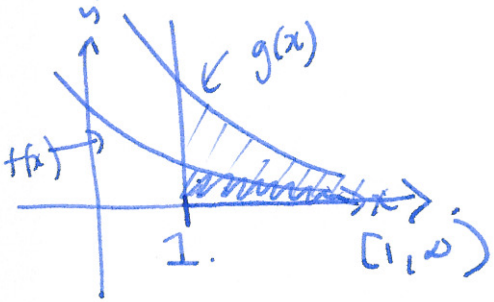
$$= \lim_{R\to 0} 2 - 2\sqrt{R} = 2$$

warning $\int_{-1}^1 \frac{1}{x}\,dx = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx = \lim_{R\to 0}\left[\ln|x|\right]_{-1}^R + \lim_{R\to 0}\left[\ln|x|\right]_R^1$



$$= \lim_{R\to 0} \ln|R| + \lim_{R\to 0} \ln|R| \qquad \text{DNE}$$

comparison test



suppose $0 \leq f(x) \leq g(x)$ on $[1, \infty)$.

if $\int_1^\infty g(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ converges.

Example $\int_1^\infty \underbrace{\frac{1}{\sqrt{x^3+1}}}_{f(x)}\,dx$.

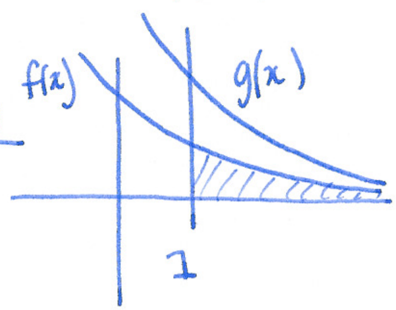
note $\sqrt{x^3+1} \geq \sqrt{x^3}$. ~~(on [1, ∞))~~.

$0 \leq \frac{1}{\sqrt{x^3+1}}^{f(x)} \leq \frac{1}{\sqrt{x^3}}^{g(x)}$. $x \geq 1$.

consider $\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \lim_{R\to\infty}\int_1^R x^{-3/2}\,dx = \lim_{R\to\infty}\left[-2x^{-1/2}\right]_1^R = \lim_{R\to\infty} -2R^{-1/2} - -2$

$= \lim_{R\to\infty} 2 - \frac{2}{\sqrt{R}} = 2 \Rightarrow \int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges. (but we don't know exact value).

other way



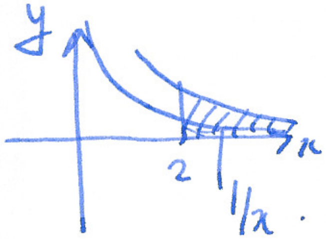
$0 \leq f(x) \leq g(x)$.

if $\int_1^\infty f(x)\,dx$ diverges $(\to\infty)$ then $\int_1^\infty g(x)$ diverges.

warning $\int_1^\infty g(x)\,dx$ diverges $\not\Rightarrow$ $\int_1^\infty f(x)\,dx$ diverges

$\int_1^\infty f(x)\,dx$ converges $\not\Rightarrow$ $\int_1^\infty g(x)\,dx$ converges

Example show $\int_2^\infty \frac{1}{x-\sqrt{x}}\,dx$ diverges.



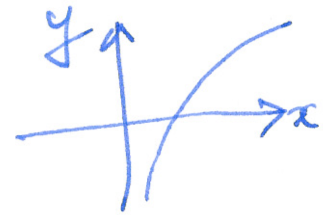
$x - \sqrt{x} < x$

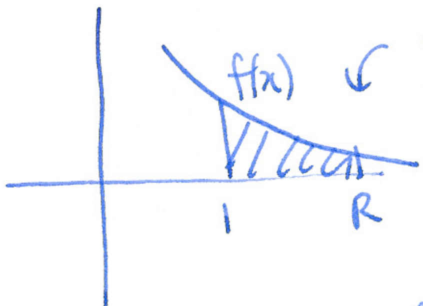
$\frac{1}{x-\sqrt{x}} > \frac{1}{x}$.

$\int_2^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_2^R \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_2^R = \lim_{R\to\infty} \ln(R) - \ln(2)$

$\to \infty$ diverges / DNE.

$\Rightarrow \int_2^\infty \frac{1}{x-\sqrt{x}}\,dx$ diverges.





+ve function

$\int_1^R f(x)\,dx \leftarrow$ positive, increasing

$\lim_{R\to\infty} \int_1^R f(x)\,dx \leftarrow$ either
- tends to a fixed positive number — converges.
- or tends to infinity — diverges.