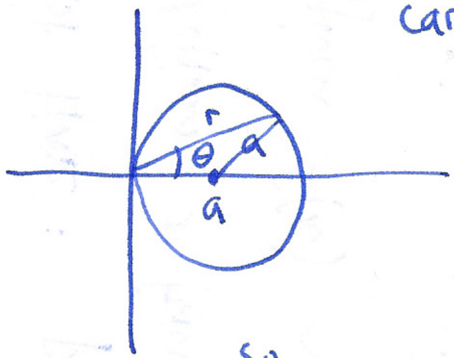
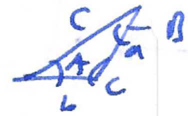


cartesian $(x-a)^2+y^2=a^2$

polar $r = 2a\cos\theta$

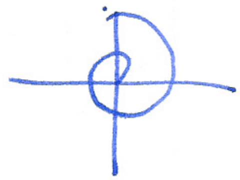


cosine rule: $a^2 = b^2+c^2-2bc\cos\theta$

so $a^2 = r^2+a^2-2ar\cos\theta \leadsto r^2 = 2ar\cos\theta$

$\leadsto r = 2ar\cos\theta$

Examples sketch $r=\theta$



$r=\sin\theta$

$r=\sin 2\theta$ etc.

can always try: $x^2+y^2=r^2$

$x=r\cos\theta$ $\quad x^2+y^2=r^2$

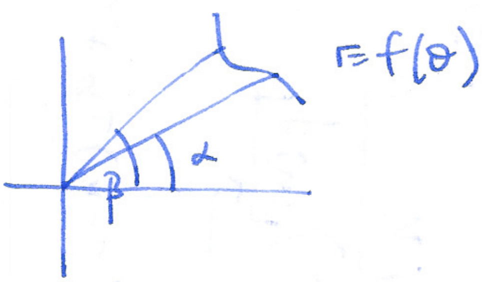
$y=r\sin\theta$ $\quad \cancel{\tan}\, y/x = \tan\theta$

$y=x^2$

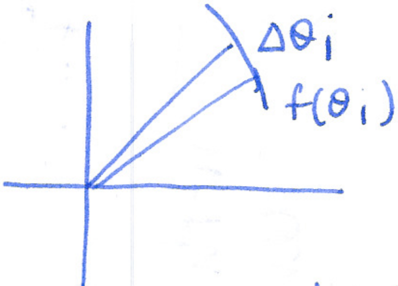
$\leadsto r\sin\theta = r^2\cos\theta$

$r = \dfrac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta.$

§11.4 Area and arc length in polars

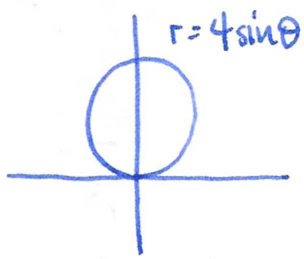


$$\text{area} = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2\, d\theta$$



$$\text{area } \Delta A \approx \frac{\pi r^2}{2\pi/\Delta\theta_i} = \frac{1}{2} r^2 \Delta\theta_i$$

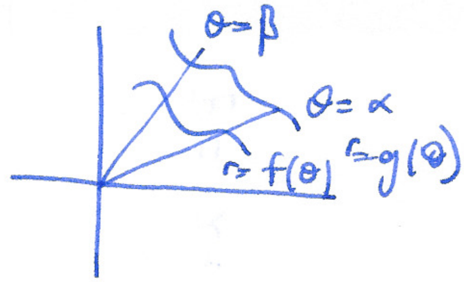
Example



$$\text{area} = \int_0^{\pi} \frac{1}{2}(4\sin\theta)^2\, d\theta = \int_0^{\pi} 8\sin^2\theta\, d\theta$$

$$= 8\int_0^{\pi} \frac{1}{2}(1-\cos 2\theta)\, d\theta = [4\theta - 2\sin 2\theta]_0^{\pi} = 4\pi.$$

area between two curves:



$$\text{area} = \frac{1}{2}\int_\alpha^\beta (g(\theta))^2 - (f(\theta))^2 \, d\theta$$

arc length $\quad r=f(\theta)$ is a parameterized curve with $x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

so $$\frac{dx}{d\theta} = f'(\theta)\cos\theta + f(\theta)(-\sin\theta)$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

arc length $$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{(f'\cos\theta - f\sin\theta)^2 + (f'\sin\theta + f\cos\theta)^2}\, d\theta$$

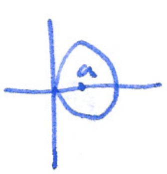
$$= \int_\alpha^\beta \sqrt{\begin{matrix} f'^2\cos^2\theta - 2ff'\sin\theta\cos\theta + f^2\sin^2\theta \\ + f'^2\sin^2\theta + 2ff'\sin\theta\cos\theta + f^2\cos^2\theta \end{matrix}}\, d\theta$$

$$= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta = \int_\alpha^\beta \sqrt{f^2 + f'^2}\, d\theta .$$

Example $\quad$ circle $\quad r = 2a\cos\theta$

$f(\theta) = 2a\cos\theta$

$f'(\theta) = -2a\sin\theta$



$$\int_{-\pi/2}^{\pi/2} \sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\, d\theta = \int_{-\pi/2}^{\pi/2} 2a\, d\theta = 2\pi a .$$