Thm If $f(x)$ is equal to a power series centered at $x=c$, with radius of convergence $R>0$, then $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, where $a_n = \frac{f^{(n)}(c)}{n!}$

useful facts: where the Taylor series converges we can:

- differentiate
- integrate
- multiply (xe^x)
- substitute (e^{x^2})

Examples xe^x, e^{-x^2}, $e^x \sin x$ etc.

Fact: works for complex numbers $x+iy$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- How to make l'Hôpitals rules problems:

$$\left.\begin{aligned} \sin(2x) &\simeq 2x - \frac{(2x)^3}{3!} + O(x^5) \\ e^{3x} &\simeq 1 + 3x + O(x^2) \end{aligned}\right\} \quad \frac{\sin(2x)}{e^{3x}-1} \simeq \frac{2x}{3x} = \frac{2}{3}.$$

- cheap error bound for e^x $(x \leq 1)$

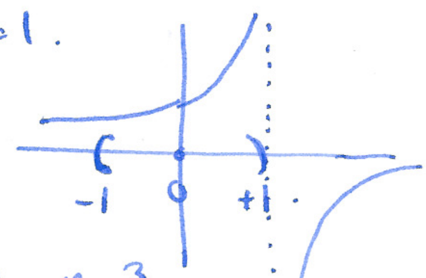
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots$$

$$\leq \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+1)!} + \cdots \quad \frac{x^{n+1}}{(1-x)(n+1)!}$$

so $e^{1/2} = 1 + 1/2 + \frac{(1/2)^2}{2!} + \cdots + \frac{(1/2)^n}{n!}$ with error $\leq \frac{1}{2^{n+2}(n+1)!}$

- **Example** $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ centered at 0, $R=1$.

defined for all $x \neq 1$ ← warning! only defined for $x \in (-1,1)$



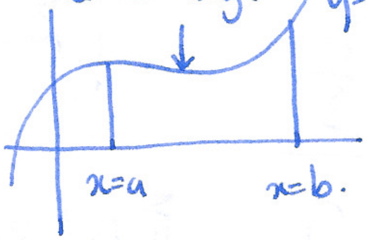
Fact: could then take power series centered at $x = -\frac{1}{2}$, $R = \frac{3}{2}$, ...

so the power series $1+x+x^2+\cdots$ can be used to define a function on $(-\infty, 1)$.

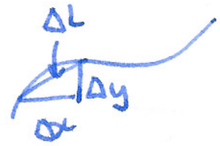
(in fact on $\mathbb{C} \setminus \{1\}$).

§8.1 Arc length and surface area

Q: how long?



A:



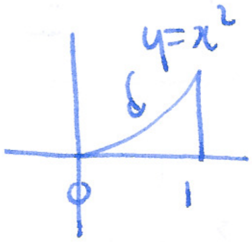
$$|\Delta L| = \sqrt{(\Delta x)^2+(\Delta y)^2}.$$

$$\text{length} = \sum_{i=1}^{n}|\Delta L_i| = \sum_{i=1}^{n}\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$$

$$= \sum_{i=1}^{n}\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$ take limit as $|\Delta x_i| \to 0$, arc length

$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Example ①



$\frac{dy}{dx}=2x$ arc length $\int_0^1 \sqrt{1+4x^2}\,dx$

(trig sub integral)

② $f(x) = \frac{1}{12}x^3+\frac{1}{x}$ on $[1,2]$

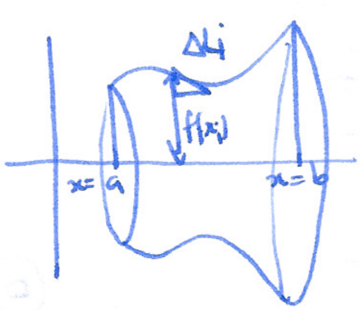
$f'(x) = \frac{1}{4}x^2 - \frac{1}{x^2}$

$$\text{arc length} = \int_1^2 \sqrt{1+\left(\tfrac{1}{4}x^2-\tfrac{1}{x^2}\right)^2}\,dx = \int_1^2\sqrt{1+\tfrac{1}{16}x^4-\tfrac{1}{2}+\tfrac{1}{x^4}}\,dx$$

$$= \int_1^2\sqrt{\tfrac{1}{16}x^4+\tfrac{1}{2}+\tfrac{1}{x^4}}\,dx = \int_1^2\sqrt{\left(\tfrac{1}{4}x^2+\tfrac{1}{x^2}\right)^2}\,dx = \int_1^2 \frac{x^2}{4}+\frac{1}{x^2}\,dx$$

$$= \left[\frac{x^3}{12}-\frac{1}{x}\right]_1^2 = \frac{8}{12}-\frac{1}{4}-\frac{1}{12}+1 = \frac{4}{3}.$$

Area of surface of revolution



$$\text{area} \approx \sum_{i=1}^{n}|\Delta L_i|\,f(x_i)\,2\pi \qquad |\Delta L_i| = \sqrt{(\Delta x_i)^2+(\Delta y_i)^2}.$$

take limit as $|\Delta x_i| \to 0$

surface area of surface of revolution $= \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx.$