Example for what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converge?

ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{2} = \frac{|x|}{2}$

so converges for $\frac{|x|}{2} < 1 \iff |x| < 2$ (so $R = 2$)

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (x-5)^n$ ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{n+1} \cdot \frac{n}{(x-5)^n} \right|$

$= \lim_{n\to\infty} \left| (x-5) \frac{n}{n+1} \right| = |x-5| < 1$ so converges for $|x-5| < 1$, i.e. $R = 1$.

Examples $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sum_{n=0}^{\infty} n! x^n$.

Thm Suppose $F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ has radius of convergence $R > 0$, then $F(x)$ is differentiable on $(a-R, a+R)$, and furthermore:

$$\left.\begin{aligned} F'(x) &= \sum_{n=1}^{\infty} a_n n (x-a)^{n-1} \\ \int F(x)\,dx &= \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1} + C \end{aligned}\right\} \text{ with } \underline{\text{same}} \text{ radius of convergence.}$$

Example ① $1 + x + x^2 + \cdots = \frac{1}{1-x}$, radius of convergence $R = 1$.

so $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

and $\int \frac{1}{1-x}\,dx = -\ln|1-x| + c = c + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

② $$\left.\begin{aligned} &1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2} \\ &\text{so } \tan^{-1}(x) = c + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}\right\} \text{ Q: radius of convergence?}$$

③ **Fact** $y = e^x$ is the (unique up to scalar multiple) solution to $\frac{dy}{dx} = y$

$\frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \cdots\right) = 1 + x + \frac{x^2}{2!} + \cdots$ so $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ $R = \infty$.

How to find a power series solution to a differential equation:

try: $y = a_0 + a_1 x + a_2 x^2 + \cdots$

Example $y' = y$: $\quad a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 + a_1 x + a_2 x^2 + \cdots$

$a_1 = a_0$, $\quad 2a_2 = a_1 = a_0 \Rightarrow a_2 = \frac{a_0}{2}$ $\quad 3a_3 = a_2 = \frac{a_0}{2} \Rightarrow a_3 = \frac{a_0}{6} = \frac{a_0}{3!}$

so $y = a_0\left(1 + x + \frac{x^2}{2!} + \cdots\right)$

Example $y'' + y' + y = 0$ with $y(0) = 1$ $y'(0) = 1$.

look for a solution $y = a_0 + a_1 x + a_2 x^2 + \cdots$ $\quad y(0) = 1 \Rightarrow a_0 = 1$

$y'(0) = 1 \Rightarrow a_1 = 1$

$y = 1 + x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = 1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = 2a_2 + 6a_3 x + \cdots$

$(y'' + y' + y) = 0 = (1 + 1 + 2a_2) + (1 + 2a_2 + 6a_3)x + (\text{---})x^2 + \cdots$ $\quad a_2 = -1$, $a_3 = -1/6$ etc.

§10.7 Taylor series

suppose $f(x)$ has a power series expansion at $x = c$, i.e.

$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$ $\quad$ Q: how do we find the a_i?

A: $f(c) = a_0$

$f'(x) = a_1 + 2a_2(x-c) + \cdots$ $\quad f'(c) = a_1$

$f''(x) = 2a_2 + 6a_3(x-c) + \cdots$ $\quad f''(c) = a_2$ 2 etc.

$$\boxed{a_n = \frac{f^{(n)}(c)}{n!}}$$

Examples

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$