Example $S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$ $\quad f(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$.

$$\int_1^\infty x^{-1/2}\,dx = \lim_{N\to\infty} \int_1^N x^{-1/2}\,dx = \lim_{N\to\infty} \left[2x^{1/2}\right]_1^N = \lim_{N\to\infty} 2\sqrt{N} - 2 \to \infty \text{ as } N\to\infty$$

so S diverges.

Thm Convergence of p-series: $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$

Proof (integral test) $\sum_{n=1}^\infty a_n$ $\quad a_n = f(n)$ $\quad f(x) = \frac{1}{x^p} = x^{-p}$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p > 1$, diverges if $p \leq 1$ $\square$.

Example $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^\infty \frac{1}{n^2}$ converges (to $\frac{\pi^2}{6}$)

$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = \sum_{n=1}^\infty \frac{1}{n^3}$ converges (to ?)

Thm **Comparison test** suppose $0 \leq a_n \leq b_n$ for all $n \geq M$

then ① if $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges

② if $\sum_{n=1}^\infty a_n$ diverges, then $\sum_{n=1}^\infty b_n$ diverges.

Example $\sum_{n=1}^\infty 2^{-n^2} = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^9} + \cdots$

note: $\frac{1}{2^{n^2}} < \frac{1}{2^n}$ geometric series, converges.

Thm **Limit comparison test** a_n, b_n **positive** series.

suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L < \infty$ exists

then
- if $L > 0$ $\sum_{n=1}^\infty a_n$ converges iff $\sum_{n=1}^\infty b_n$ converges
- if $L = 0$ $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges.

<u>Proof</u> (sketch) case $L > 0$, then $\frac{a_n}{b_n} \to L$, so $0 < \frac{a_n}{b_n} < R$ for some $R > L$, so $0 < a_n < R b_n$

comparison test: $\sum b_n$ converges $\Rightarrow$ $R \sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges.

similarly if $\frac{b_n}{a_n} \to \frac{1}{L}$, then $0 < \frac{b_n}{a_n} < R'$ for some $\frac{1}{L} < R'$

so $0 < b_n < R' a_n$, so $\sum a_n$ converges $\Rightarrow$ $\sum b_n$ converges.

(if $L = 0$ only one direction works) $\square$.

<u>Example</u> show $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges (for large n, $a_n \sim \frac{1}{n^2}$).

compare with $b_n = \frac{1}{n^2}$ $\quad \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n^2}{n^4 n - 1}}{1/n^2} = \frac{n^4}{n^4 - n - 1} = 1$.

so $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{n^2}{n^4 - n - 1}$ converges, by limit comparison test.

<u>Example</u> does $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2 - 9}}$ converge? compare with $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2-9}}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-9}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 - 9/n^2}} = 1$$

so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-9}}$ diverges, by limit comparison test.

<u>§10.4 Absolute and conditional convergence</u>

<u>Q</u>: what about $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$? (*)

<u>Defn</u> A series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

(*) is absolutely convergent

<u>Example</u> $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ <u>not</u> absolutely convergent

<u>Thm</u> Absolute convergence $\Rightarrow$ convergent.

Proof $\quad 0 \le a_n + |a_n| \le 2|a_n|$

$\sum_{n=1}^{\infty} 2|a_n| = 2\sum_{n=1}^{\infty} |a_n|$ converges (by absolute convergence)

$\Rightarrow \sum_{n=1}^{\infty}(a_n+|a_n|)$ converges by comparison test

then $\sum_{n=1}^{\infty} a_n+|a_n|-|a_n| = \sum_{n=1}^{\infty} a_n+|a_n| \cancel{+} - \sum_{n=1}^{\infty} |a_n|$

converges $\Leftarrow$ converges $\quad$ converges

$= \sum_{n=1}^{\infty} a_n$ converges $\square$.

Q: what about $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$?

Def $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u> if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty}|a_n|$ does not converge.

Thm (Alternating series test) Let a_n be a positive, decreasing sequence with $a_n \to 0$. Then $\underbrace{\sum_{n=1}^{\infty}(-1)^{n+1}a_n}_{S}$ converges. Furthermore $0 \le S \le a_1$

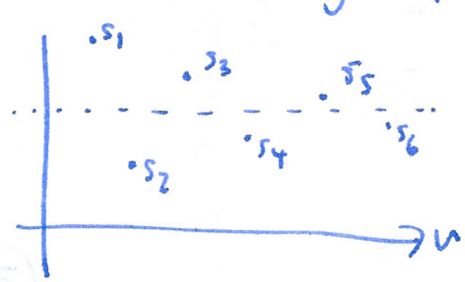
and $S_{2n} \le S \le S_{2n+1}$ for all n.

Proof even partial sums: $\quad S_{2n} = \underbrace{a_1-a_2}_{\ge 0} + \underbrace{a_3-a_4}_{\ge 0} + \cdots + \underbrace{a_{2n-1}-a_{2n}}_{\ge 0}$

↑ positive increasing sequence

odd partial sums: $\quad S_{2n+1} = a_1 - (\underbrace{a_2-a_3}_{\ge 0}) - (\underbrace{a_4-a_5}_{\ge 0}) - \cdots - (\underbrace{a_{2n}-a_{2n+1}}_{\ge 0})$

↑ decreasing sequence



Furthermore $S_{2n} = a_1 - (a_2-a_3) - \cdots - a_{2n}$

so $S_{2n} \le a_1$ for all n

so S_{2n} is an increasing sequence, bounded above

so $\lim_{n\to\infty} S_{2n}$ exists.

similarly $\lim_{n\to\infty} s_{2n+1}$ exists. $s_{2n+1} = (a_1-a_2)+(a_3-a_4)+\dots+(a_{2n-1}-a_{2n})+a_{2n+1} \geq (a_1-a_2)$.

note: $\lim_{n\to\infty} s_{2n}-s_{2n+1} = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} -a_{2n+1} = 0$ $\square$.

Example show $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots$ converges (alternating harmonic series)
use alternating series test: $a_n = \frac{1}{n}$ check: positive, decreasing, $a_n \to 0$
so $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. $\square$.
So $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\dots$ is conditionally convergent (Fact: $= \ln(2)$).

§10.5 Ratio and root tests.

Fact $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$ Q: how do we know this converges?
(e.g. use comparison test $n! = 1\cdot 2\cdot 3 \dots (n-1)n. > (n-1)^2$ so $\frac{1}{n!} < \frac{1}{(n-1)^2}$).

Thm Ratio test: (a_n) sequence and suppose $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.
then ① if $\rho < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely
② if $\rho > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges
③ if $\rho = 1$ no information.

Proof if $\rho < 1$, then there is a number $\rho < r < 1$, and a number N
s.t. $\left|\frac{a_{n+1}}{a_n}\right| < r$ for all $n \geq N$, so $|a_{N+1}| < r|a_N|$
$|a_{N+2}| < r|a_{N+1}| < r^2|a_N|$ etc.
so $\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} |a_N| r^n \leq \frac{|a_N|}{1-r}$ so converges by comparison test with geometric series.
if $\rho > 1$, then there is $\rho > r > 1$ and N s.t.
$\frac{|a_{n+1}|}{|a_n|} > r$ for all $n \geq N$, so $a_n \not\to 0 \Rightarrow \sum a_n$ diverges. $\square$.

Example ① show $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ratio test: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$

(2) show $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3} < 1$.

<u>Bad example</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{1}{1+1/n}\right)^2 = 1$

<u>Thm</u> <u>Root test</u> (a_n) sequence, suppose that $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists

(1) if $L < 1$ then $\sum a_n$ converges absolutely

(2) if $L > 1$ then $\sum a_n$ diverges

(3) if $L = 1$ no information

<u>Example</u> $\sum_{n=1}^{\infty} \left(\frac{n}{2n+3}\right)^n$

<u>§10.6 Power series</u>

<u>Defn</u> A power series centered at $x=a$ is an infinite sum of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

<u>note</u>, if the series converges, this gives a function of x.

the series always converges for $x=a$! $F(a) = a_0$

<u>Thm</u> <u>Radius of convergence</u> Let $F(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$, then

(1) $F(x)$ converges only for $x=a$ $(R=0)$

or (2) $F(x)$ converges for all x $(R=\infty)$

or (3) there is an $R>0$ s.t. $F(x)$ converges for all $|x-a|<R$ and diverges for all $|x-a|>R$. May or may not converge for $|x-a|=R$, i.e. $a-R, a+R$. R is called the <u>radius of convergence</u>.