$s_N = 1 - \frac{1}{2^N}$ $\quad \lim_{N\to\infty} s_N = \lim_{N\to\infty} 1 - \frac{1}{2^N} = 1$ $\quad$ so $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1.$

<u>General case</u> $\quad c + cr + cr^2 + cr^3 + \cdots = \sum_{n=0}^{\infty} cr^n$

$$s_N = c + cr + cr^2 + \cdots + cr^N$$
$$rs_N = cr + cr^2 + \cdots + cr^N + cr^{N+1}$$

$$s_N - rs_N = c - cr^{N+1}$$

$s_N(1-r) = c(1-r^{N+1})$ $\quad s_N = \frac{c(1-r^{N+1})}{(1-r)}$ $\quad \lim_{N\to\infty} s_N = \frac{c}{1-r}$ if $|r|<1$

<u>otherwise does not converge</u>

<u>Special case</u> (Telescoping)

<u>Example</u> $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ $\quad$ so $\quad s_1 = 1 - \frac{1}{2}.$

$$s_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}.$$
$$s_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

so $s_N = 1 - \frac{1}{N+1}$ $\quad \lim_{N\to\infty} s_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1.$

<u>Example</u> (harmonic series) $\quad \underbrace{1}_{\geq 1} + \underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{\geq \frac{1}{2}} + \cdots$

diverges!

<u>Q</u>: when does a series converge? $\quad$ A: dunno

<u>Tools</u>

- <u>The</u> Divergence test $\quad$ If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ <u>diverges</u>

<u>Warning</u> $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ converges. $\quad$ <u>Recall</u>: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges.

<u>Rules for series, AKA infinite sums</u>

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series then $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

$$\sum_{n=1}^{\infty}(a_n-b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c a_n = c\sum_{n=1}^{\infty} a_n.$$

observation $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{(\sqrt{2})^{2n}}$

§10.3 Positive series

positive series : $\sum_{n=1}^{\infty} a_n = a_1+a_2+a_3+\cdots$ all $a_n \geq 0$

note : the sequence S_N is an increasing sequence : $S_{N+1} = S_N + \underbrace{a_{n+1}}_{\geq 0}$

recall : an increasing sequence with an upper bound converges.

Thm Let $S = \sum_{n=1}^{\infty} a_n$ be a positive series. Then exactly one of the following occurs: ① the partial sums are bounded above, and S converges

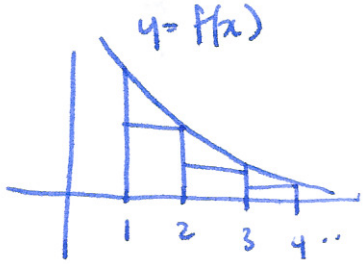
② the partial sums are not bounded above, and S diverges

Thm Integral test Let $a_n = f(n)$ $f(x)$ $\left.\begin{array}{l}\text{positive}\\ \text{decreasing}\\ \text{cts}\end{array}\right\}$ for $x \geq 1$

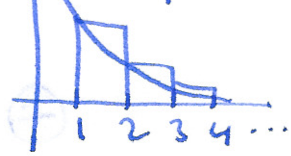
then ① if $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

② if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges

Proof



$$S_N - a_1 = a_2+a_3+\cdots+a_N \leq \int_1^N f(x)\,dx$$



$$S_N = a_1+a_2+\cdots+a_N \geq \int_1^N f(x)\,dx. \quad \square.$$