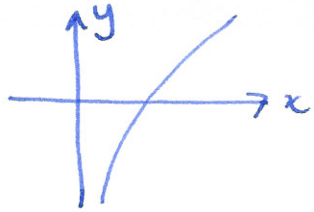
Example $y = \ln(x)$ at $x = 1$



$f(x) = \ln(x)$ $\quad f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad f'(1) = 1$

$f''(x) = -x^{-2}$ $\quad f''(1) = -1$

$f^{(3)}(x) = 2x^{-3}$ $\quad f^{(3)}(1) = 2$

$f^{(4)}(x) = -6x^{-4}$ $\quad f^{(4)}(1) = -3!$

$f^{(5)}(x) = 4!\,x^{-5}$ $\quad f^{(5)}(1) = 4!$

$f^{(k)}(x) = (k-1)!(-1)^{k+1}x^{-k}$ $\quad f^{(k)}(1) = (-1)^{k+1}(k-1)!$

so $T_n(x) = 0 + 1.(x-1) + (-1)\frac{(x-1)^2}{2!} + \frac{2}{3!}(x-1)^3 + \cdots + (-1)^{n+1}\frac{(n-1)!}{n!}(x-1)^n$

Taylor series is $T(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$

__warning__ this may not converge! $T(3) = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \cdots$

__Q__: how good are the Taylor approximations?

__Thm__ (error bound) $f(x)$ function, $f^{(n)}(x)$ exists and is cts.

Let K be an upper bound for $|f^{(n+1)}(u)|$ for all u in $[a,x]$.

then $|T_n(x) - f(x)| \le \frac{K|x-a|^{n+1}}{(n+1)!}$

__Example__ $f(x) = e^x$. Find $T_4(x)$ at $x=0$, then find error bound for $T_4(1)$.

$\left.\begin{array}{ll} f(x) = e^x & f(0) = 1 \\ \vdots & \vdots \\ f^{(4)}(x) = e^x & f^{(4)}(0) = 1 \end{array}\right\}$ $T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + x^4/4!$

$T_4(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \approx 2.70833$.

__error bound__ : need to find K s.t. $|f^{(n+1)}(x)| \le K$ for all $x \in [0,1]$

i.e. want $|e^x| \le K$ for all $x \in [0,1]$, can choose $K = 3$.

so $|e - T_4(1)| \le \frac{3 \cdot 1^{n+1}}{(n+1)!} = \frac{3}{120}$

quick upper bound for e:

$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

$\le 1 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 3$.

§10.1 Sequences

Def^n^ A sequence is a list of numbers indexed by $\mathbb{N}$ ← positive integers.

Examples $1, 2, 3, 4, \dots$ notation: $a_1, a_2, a_3, \dots$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ or $(a_n)_{n \in \mathbb{N}}$ or (a_n)

$1, 1, 1, 1, \dots$

$1, 27, \pi, \sqrt{2}, \dots$ ↑ a_n is the n-th number in the sequence

Q: what is not a sequence? a single number, a set of numbers, a function ...

sometimes (but not always) we can give a sequence by a formula

Examples $(a_n)_{n \in \mathbb{N}}$ $a_n = n$ $(n)_{n \in \mathbb{N}}$ $1, 2, 3, 4, \dots$

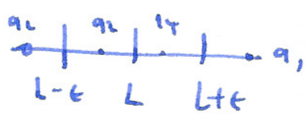
$(a_n)_{n \in \mathbb{N}}$ $a_n = \frac{1}{n}$ $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

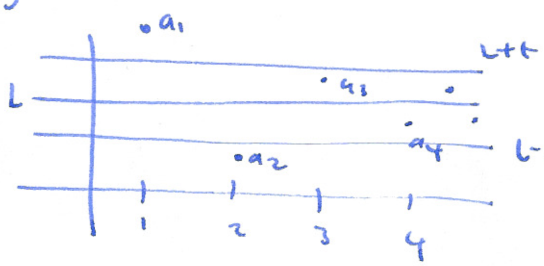
$a_n = \frac{1}{1+n^2}$ $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \dots$

Example (recursive def^n^)

$a_{n+2} = a_n + a_{n+1}$ $a_1 = 1, a_2 = 1$ gives: $1, 1, 2, 3, 5, 8, 13, \dots$ (Fibonacci sequence)

Def^n^ A sequence (a_n) converges to L if for every $\epsilon > 0$ there is an N s.t. $|a_n - L| \leq \epsilon$ for all $n \geq N$





Notation $\lim_{n \to \infty} a_n = L$ or $a_n \to L$

Examples $(a_n) = \left(\frac{1}{n}\right)$ $\lim_{n \to \infty} \frac{1}{n} = 0$

Proof given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$. if $n > N$ then $\frac{1}{n} < \frac{1}{N} < \epsilon$, as required □

Special case: sequence defined by a function $f(x)$, i.e. $a_n = f(n)$

Thm If $\lim_{x \to \infty} f(x) = L$ then $\lim_{n \to \infty} f(n) = L$

Q: is the converse true A: no.

Example $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$ $a_n = \frac{n-1}{n}$ $f(x) = \frac{x-1}{x} = 1 - \frac{1}{x}$

$\lim_{x \to \infty} 1 - \frac{1}{x} = 1 \Rightarrow \lim_{n \to \infty} a_n = 1$.

<u>Example</u> (geometric series) $a_n = r^n$

e.g. $2, 4, 8, 16, \ldots \quad a_n = 2^n$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \quad a_n = \frac{1}{2^n}$

$1, 1, 1, \ldots \quad a_n = 1^n$

<u>Fact</u> $\lim_{n\to\infty} r^n = \begin{cases} \infty & r > 1 \\ 1 & r = 1 \\ 0 & |r| < 1 \\ \text{DNE} & r \leq -1 \end{cases}$

rules for limits of sequences: same as rules for limits of functions

suppose $a_n \to L$ and $b_n \to M$, then

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$
- $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = LM$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\lim_{n\to\infty} a_n\right) / \left(\lim_{n\to\infty} b_n\right) = L/M$ as long as $M \neq 0$.
- $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n = cL$, c constant, does not depend on n

<u>squeeze Thm</u> If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$,

then $\lim_{n\to\infty} b_n = L$

<u>Example</u> $\lim_{n\to\infty} \frac{R^n}{n!} = 0$ for any R

<u>Proof</u> there is an integer M s.t. $M \leq R \leq M+1$

$$0 \leq \frac{R^n}{n!} = \underbrace{\frac{R}{1} \cdot \frac{R}{2} \cdots \frac{R}{M}}_{\text{call this } A} \cdot \underbrace{\frac{R}{M+1} \cdots \frac{R}{n-1}}_{\leq 1} \cdot \frac{R}{n} \leq A \cdot \frac{R}{n}$$

so $0 \leq \frac{R^n}{n!} \leq A \cdot \frac{R}{n}$ $\quad \lim_{n\to\infty} 0 = 0 \quad \lim_{n\to\infty} A\frac{R}{n} = 0 \Rightarrow \lim_{n\to\infty} \frac{R^n}{n!} = 0$

Thm If $f(x)$ is cts (at L) and $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$

important f cts at L Bad example $f(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$ then $\frac{1}{n} \to 0 \not\to$ but $f(\frac{1}{n}) = 1 \to 1$ ✗.

Example find $\lim_{n\to\infty} e^{n/n+1}$

start with $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$ then $\lim_{n\to\infty} e^{n/n+1} = e^{\lim_{n\to\infty} n/n+1} = e^1 = e$

Defn A sequence (a_n) is
- bounded above if $a_n \le M$ for all n
- bounded below if $L \le a_n$ for all n
- bounded if $L \le a_n \le M$ for all n

Thm Convergent subsequences are bounded

Warning: bounded subsequences need not converge.

Example: $0, 1, 0, 1, \ldots$ $a_n = \frac{1-(-1)^n}{2}$.

Thm Bounded monotonic sequences converge.
- if (a_n) is increasing and $a_n \le M$, then $a_n \to l \le M$.
- if (a_n) is decreasing and $L \le a_n$, then $a_n \to l \ge L$

Example $a_n = \frac{1}{n}$, show decreasing, want $a_n \ge a_{n+1}$

$n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$, lower bound $L = -1$, $\lim_{n\to\infty} \frac{1}{n} = l \ge -1$.

Example show $a_n = \sqrt{n+1} - \sqrt{n}$ decreasing and bounded below

note: $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n}$ as $\sqrt{x}$ monotonic $\Rightarrow a_n \ge 0$

so can choose lower bound $L = 0$

decreasing: consider $f(x) = \sqrt{x+1} - \sqrt{x} = (x+1)^{1/2} - x^{1/2}$

$f'(x) = \frac{1}{2}(x+1)^{-1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}\right)$

claim $f'(x) < 0$: $x+1 > x$

$\sqrt{x+1} > \sqrt{x}$

$\frac{1}{\sqrt{x+1}} < \frac{1}{\sqrt{x}}$ so $f'(x) < 0 \Rightarrow f(x)$ decreasing $\square$.

§10.2 Series

Defn A series is an infinite sum $a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$

Examples $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ $\quad \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ $\quad 1+1+1+\cdots$ $\quad 1-1+1-1+\cdots$

Defn The N-th partial sum $S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n$

Defn The sum of the infinite series is defined to be the limit of partial sums, if this limit exists.

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

if $\lim_{N\to\infty} S_N = S$, then we say $\sum_{n=1}^{\infty} a_n$ converges and write $\sum_{n=1}^{\infty} a_n = S$

Examples ① $1+1+1+\cdots$ $\quad S_N = \underbrace{1+1+\cdots+1}_{N} = N$ $\quad \lim_{N\to\infty} N = \infty$, so $\sum_{n=1}^{\infty} 1$ does not converge

② $1-1+1-1+1-\cdots$ $\quad s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ s_4 = 0, \ldots$

$(s_N) = 1, 0, 1, 0, 1, \ldots$ does not converge.

Warning can't necessarily re-arrange terms : $(1-1)+(1-1)+\cdots = 0$

$1(-1+1)+(-1+1)+\cdots = 1$

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} \qquad a_n = \frac{1}{2^n}$$

$s_1 = \frac{1}{2} = 1 - \frac{1}{2}$.

$s_2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$

$s_3 = \frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$

$$\left.\begin{array}{l} S_N = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots + \frac{1}{2^N}. \\ \frac{1}{2}S_N = \qquad \frac{1}{4}+\frac{1}{8}+\cdots + \frac{1}{2^N} + \frac{1}{2^{N+1}} \end{array}\right\} \begin{array}{l} S_N - \frac{1}{2}S_N = \frac{1}{2} - \frac{1}{2^{N+1}} \\ \frac{1}{2}S_N = \frac{1}{2} - \frac{1}{2^{N+1}} \end{array}$$