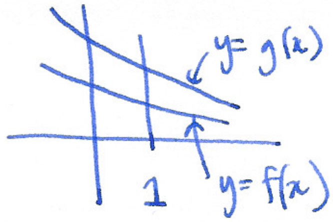
Comparison test



suppose $0 \le f(x) \le g(x)$ on $[1, \infty)$

if $\int_1^\infty g(x)\,dx$ converges, then $\int_1^\infty f(x)\,dx$ converges

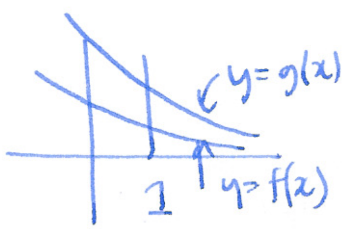
Example $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ note $\sqrt{x^3+1} \ge \sqrt{x^3}$ on $[1, \infty)$.

so $0 \le \frac{1}{\sqrt{x^3+1}} \le \frac{1}{\sqrt{x^3}}$

consider $\int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \int_1^\infty x^{-3/2}\,dx = \lim_{R\to\infty} \left[-2x^{-1/2}\right]_1^R = \lim_{R\to\infty} \frac{-2}{\sqrt{R}} + 2 = 2 < \infty$

so $\int_1^\infty \frac{1}{\sqrt{x^3+1}}\,dx$ converges (but don't know exact value!)

other way



$0 \le f(x) \le g(x)$

if $\int_1^\infty f(x)\,dx$ diverges $\Rightarrow \int_1^\infty g(x)\,dx$ diverges

note: $\int_1^\infty g(x)\,dx$ diverges $\rightsquigarrow$ no information about f!

$\int_1^\infty f(x)\,dx$ converges $\rightsquigarrow$ no information about g!

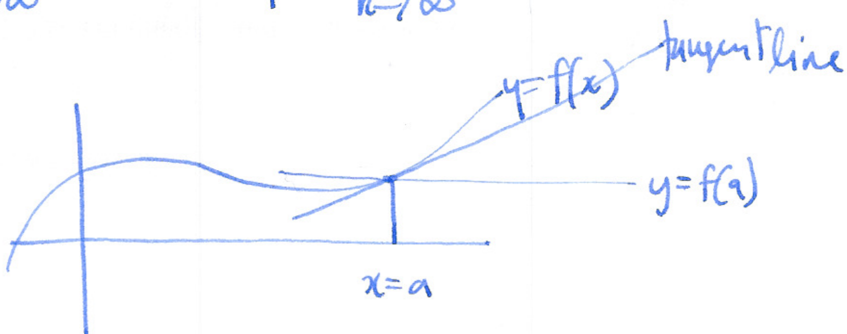
Example show $\int_2^\infty \frac{1}{x-\sqrt{x}}\,dx$ diverges

$x - \sqrt{x} < x$

$\frac{1}{x-\sqrt{x}} > \frac{1}{x}$

$\int_2^\infty \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_1^R = \lim_{R\to\infty} \ln|R| - 0 \to \infty.$

§8.4 Taylor Polynomials



Approximating functions at $x = a$.

0-th approx: $f(x) \simeq f(a)$

1-st approx: (straight line): $f(x) \simeq f(a) + f'(a)(x-a)$ (tangent line)

2nd approx?

2nd approx: (quadratic) ? how do we find these ?
3rd approx: (cubic)

tangent line: unique line with same value as $f(x)$ at $x=a$ $(f(a))$.
same slope as $f(x)$ at $x=a$ $(f'(a))$.

quadratic: same value as $f(x)$ at $x=a$ $f(a)$
same slope as $f(x)$ at $x=a$ $f'(a)$ $y = ax^2+bx+c$.
same 2nd derivative as $f(x)$ at $x=a$ $f''(a)$

cubic: same first 3 derivatives.

etc.

<u>quadratic</u> $T_2(x) = ax^2+bx+c$

better: $T_2(x) = a_0 + a_1(x-a) + a_2(x-a)^2$

deg 0: $T_2(a) = a_0 \qquad = f(a)$ (same value)

deg 1: $T_2'(x) = a_1 + 2a_2(x-a)$

$T_2'(a) = a_1 \qquad = f'(a)$ (same slope)

deg 2: $T_2''(x) = 2a_2$

$T_2''(a) = 2a_2 \qquad = f''(a)$ (same 2nd derivative)

so $T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

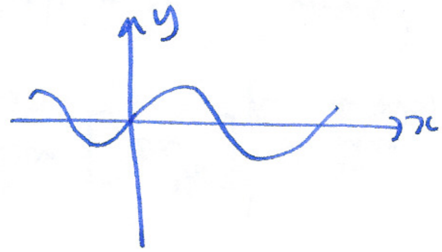
<u>In general</u> $T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n$

differentiate k times: $T_n^{(k)}(x) = k!\,a_k +$ (stuff with $(x-a)$ factors)

$T_n^{(k)}(a) = k!\,a_k = f^{(k)}(a)$ so $a_k = \frac{1}{k!}f^{(k)}(a)$.

so $T_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$= \sum_{k=0}^{n} \frac{f^{(n)}(a)}{k!}(x-a)^k$



Example ① $y = \sin x$, at $a = 0$

$f(x) = \sin(x) \qquad f(0) = 0$

$f'(x) = \cos(x) \qquad f'(0) = 1$

$f''(x) = -\sin(x) \qquad f''(0) = 0$

$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$

so $T_4(x) = 0 + 1 \cdot x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4$

$= x - \frac{x^3}{3!}$

② $y = \cos(x)$, at $x = 0$

$f(x) = \cos(x) \qquad f(0) = 1$

$f'(x) = -\sin(x) \qquad f'(0) = 0$

$f''(x) = -\cos(x) \qquad f''(0) = -1$

$f^{(3)}(x) = \sin(x) \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x) \qquad f^{(4)}(0) = 1$

so $T_4(x) = 1 + 0 \cdot x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4$

$T_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

③ $y = e^x$ at $x = 0$

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

Sneak preview: what if we continue the pattern for ever? not polynomial, an **infinite sum** $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ ← called a Taylor series

Fact $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

Remark $e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$

$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right) = \cos\theta + i\sin\theta$.