Linear and quadratic factors

$$\int \frac{1}{x(1+x^2)}\,dx \qquad \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$1 = 0x^2 + 0x + 1 = x^2(A+B) + x(C) + A \qquad \left.\begin{aligned} A+B&=0 \\ C&=0 \\ A&=1 \end{aligned}\right\} B=-1$$

$$= \int \frac{1}{x} + \frac{-x}{x^2+1}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1| + c$$

repeated quadratic factors

$$\frac{1}{(x+a)(x+b)^2(x^2+c^2)^2} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{(x+b)^2} + \frac{Dx+E}{x^2+c^2} + \frac{Fx+G}{(x^2+c^2)^2}$$

solve for $A, B, \ldots$

moral: every $\frac{P(x)}{Q(x)}$ can be integrated $\square$.

§7.6 Strategies for integration

tools:

- rewrite expressions, e.g.

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2+x+1$$

$$\frac{x-x^3}{\sqrt{x}} = x^{1/2} - x^{5/2}.$$

  - trig identities
  - partial fractions

- substitutions
- parts
- trig ~~identit~~ integrals.

Examples ① $\int x^3\sqrt{1+x^2}\,dx$ try $u = 1+x^2$, $\frac{du}{dx} = 2x$

$$\int x^3 u^{1/2} \frac{dx}{du}\,du = \int x^3 u^{1/2} \frac{1}{2x}\,du$$

$$= \int \frac{1}{2}x^2 u^{1/2}\,du = \int \frac{1}{2}(u-1)u^{1/2}\,du = \frac{1}{2}\int u^{3/2} - u^{1/2}\,du = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + c$$

$$= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + c$$

② $\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx$ try $u = \sqrt{x}$, $u = \sqrt{x}+1$, $\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$$\int \frac{1}{\sqrt{u}} \cdot 2x^{1/2}\,du = 2\int \frac{u-1}{\sqrt{u}}\,du = \cancel{2\int u^{1/2}\,du}$$

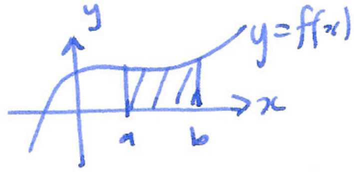
$$= 2\int u^{1/2} - u^{-1/2}\,du = \frac{4}{3}u^{3/2} - 4u^{1/2} + c$$

$= \frac{4}{3}(\sqrt{x}+1)^{3/2} - 4(\sqrt{x}+1)^{1/2} + c$

③ $\int \sqrt{x^2+2x+2}\,dx$ complete the square $\int \sqrt{(x+1)^2+1}\,dx$, trig sub...

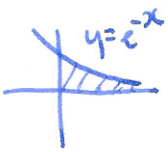
§7.7 Improper integrals

recall $\int_a^b f(x)\,dx$ = area under the curve



Q: what about integrals over infinite intervals?

Example

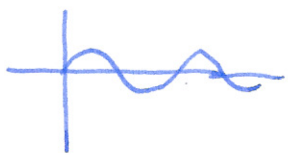


$\int_0^\infty e^{-x}\,dx$? note $\int_0^R e^{-x}\,dx = \left[-e^{-x}\right]_0^R = -e^{-R} + e^0 = 1 - e^{-R}$

Def^n $\int_0^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$ if this limit exists. Otherwise DNE/undefined

warning: sometimes the limit doesn't exist

Example $\int_0^\infty \sin(x)\,dx$



$\int_0^R \sin(x)\,dx = \left[-\cos(x)\right]_0^R = 1 - \cos(R)$

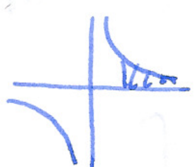
$\lim_{R\to\infty} 1 - \cos(R)$ DNE.

Example ① $\int_1^\infty \frac{1}{x^2}\,dx$

$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^R = -\frac{1}{R} + 1$

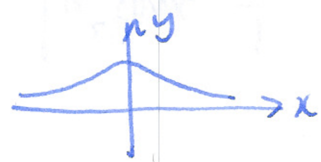
$\lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx = \lim_{R\to\infty} -\frac{1}{R} + 1 = 1$

② $\int_1^\infty \frac{1}{x}\,dx$



$\lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx = \lim_{R\to\infty} \left[\ln|x|\right]_1^R = \lim_{R\to\infty} \ln|R| \to \infty$ as $R\to\infty$.

Doubly infinite integrals $\int_{-\infty}^\infty f(x)\,dx$, $f(x)$ cts.



Def^n (f cts) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x) + \int_a^\infty f(x)\,dx$, provided each limit exists.

Example $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \cancel{\lim_{R\to\infty}\left[\tan^{-1}(x)\right]_{-R}^{0}}$

$= \lim_{R\to\infty} \int_{-R}^{0} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_{0}^{R} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_{-R}^{0} + \lim_{R\to\infty} \left[\tan^{-1}(x)\right]_{0}^{R}$

$= \lim_{R\to\infty} 0 + \tan^{-1}(R) + \lim_{R\to\infty} \tan^{-1}(R) - 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

<u>warning</u> $\int_{-\infty}^{\infty} f(x)\,dx \neq \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$

<u>Example</u> $\int_{-\infty}^{\infty} x\,dx$ DNE but $\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left[\tfrac{1}{2}x^2\right]_{-R}^{R} = \lim_{R\to\infty} 0 - 0.$

<u>Example</u> $\int_{0}^{\infty} xe^{-x}\,dx = \lim_{R\to\infty} \int_{0}^{R} xe^{-x}\,dx$ $\qquad \int uv'\,dx = uv - \int u'v\,dx$

$= \lim_{R\to\infty} \left[-xe^{-x}\right]_{0}^{R} - \lim_{R\to\infty} \int_{0}^{R} -e^{-x}\,dx = \underbrace{\lim_{R\to\infty} -Re^{-R}}_{=0} + \lim_{R\to\infty} \left[-e^{-x}\right]_{0}^{R}$

$= \lim_{R\to\infty} -e^{-R} + 1 = 1$

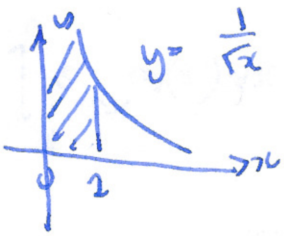
<u>Example</u> when does $\int_{1}^{\infty} \frac{1}{x^p}\,dx$ exist $\qquad$ $p=1$ no, $p=2$ yes.

$\lim_{R\to\infty} \int_{1}^{R} x^{p}\,dx = \lim_{R\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_{1}^{R} = \lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}$

$\lim R^{-p+1} = 0$ if $p > 1$

$\qquad\quad = \infty$ if $p < 1$.

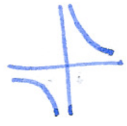
<u>integrals with discontinuities</u>



$\int_{0}^{1} \frac{1}{\sqrt{x}}\,dx$ is an improper integral! $f(0)$ not defined.

$= \lim_{R\to 0} \int_{R}^{1} \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0} \left[2x^{1/2}\right]_{R}^{1} = \lim_{R\to\infty} 2 - 2\sqrt{R} = 2$

<u>warning</u> $\int_{-1}^{1} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-1}^{1} = \ln|1| - \ln|1| = 0$ <u>wrong</u>!



$\frac{1}{x}$ discontinuous on $[-1,1]$ so need $\int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$ DNE.