§7.6 Partial fractions

aim : evaluate $\int \frac{P(x)}{Q(x)}\,dx$ $P(x), Q(x)$ polynomials.

recall : $\int \frac{1}{x+a}\,dx = \ln|x+a| + c$

special case : $\deg(P) < \deg(Q)$ and $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i all distint real numbers.

then $\frac{P(x)}{Q(x)} = \frac{A_1}{x+a_1} + \frac{A_2}{x+a_2} + \dots + \frac{A_n}{x+a_n}$ A_i numbers

Example $\int \frac{x}{x^2-1}\,dx$ $x^2-1 = (x+1)(x-1)$

$$\frac{x}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x-1)(x+1)} = \frac{x(A+B)+B-A}{x^2-1}$$

$$\left.\begin{aligned} A+B &= 1 \quad ① \\ -A+B &= 0 \quad ② \end{aligned}\right\} \quad ①+②: \ 2B = 1 \quad B = \tfrac{1}{2}, \ A = \tfrac{1}{2}$$

$$\int \frac{x}{x^2-1}\,dx = \int \frac{1/2}{x+1} + \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$$

special case : as above but $\deg(P) \geq \deg(Q)$ $\rightsquigarrow$ long division

Example $\int \frac{x^3}{x^2-1}\,dx$

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & x^3 - x \\ \hline & x \end{array}$$

x ← remainder

$x^3 = (x^2-1)x + x$

so $\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}$

so $\int \frac{x^3}{x^2-1}\,dx = \int x + \frac{x}{x^2-1}\,dx = \frac{1}{2}x^2 + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c$.

special case : $\frac{P(x)}{Q(x)}$ $Q(x) = (x+a_1)(x+a_2)\cdots(x+a_n)$

a_i not all distint, i.e. repeated roots.

Example $\dfrac{x+1}{(x-1)(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ A, B, C numbers

in general: $\dfrac{P(x)}{Q(x)}$ if $Q(x) = (x+a_1)^{n_1}(x+a_2)^{n_2} \dots (x+a_k)^{n_k}$

then $\dfrac{P(x)}{Q(x)} = \dfrac{A_{1,1}}{x+a_1} + \dfrac{A_{1,2}}{(x+a_1)^2} + \dots + \dfrac{A_{1,n_1}}{(x+a_1)^{n_1}} + \dfrac{A_{2,1}}{x+a_2} + \dots + \dfrac{A_{k,n_k}}{(x+a_k)^{n_k}}$

Example $\dfrac{x+1}{(x-1)(x+2)^2} = \dfrac{A(x+2)^2 + B(x-1)(x+2) + C(x-1)}{(x-1)(x+2)^2}$

① equate coefficients: 3 equations in 3 unknowns

② true for all $x \Rightarrow$ true for any particular x

$x=-2$: $-1 = C(-3)$ $\Rightarrow C = 1/3$

$x=1$: $2 = A.9$ $\Rightarrow A = 2/9$

$x=0$: $1 = 4A - 2B - C$ $\Rightarrow 2B = 4A - C - 1 = 8/9 - \frac{1}{3} - 1 = -4/9$

So $\displaystyle\int \frac{x+1}{(x-1)(x+2)^2}\,dx = \int \frac{2/9}{x-1} + \frac{-2/9}{x+2} + \frac{1/3}{(x+2)^2}\,dx = \frac{2}{9}\ln|x-1| - \frac{2}{9}\ln|x+2| - \frac{1}{3}\frac{1}{x+2} + c$

special case: irreducible quadratic factors

$x^2+1 = (x-i)(x+i) \neq (x+a_1)(x+a_2)$ a_i real.

note: $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

Q: $\displaystyle\int \frac{1}{x^2+x+1}\,dx$? A: complete the square $x^2+x+1 = (x+\frac{1}{2})^2 + 3/4$

$\displaystyle = \int \frac{1}{3/4 + (x+1/2)^2}\,dx$ sub $u = x + 1/2$, $\frac{du}{dx} = 1$ $\displaystyle\int \frac{1}{3/4 + u^2}\,du = \frac{4}{3}\int \frac{1}{1+\left(\frac{2u}{\sqrt{3}}\right)^2}\,du$

sub $v = \frac{2u}{\sqrt{3}}$ $\frac{dv}{du} = \frac{2}{\sqrt{3}}$ $\displaystyle = \frac{4}{3}\int \frac{1}{1+v^2}\cdot\frac{\sqrt{3}}{2}\,dv = \frac{2}{\sqrt{3}}\tan^{-1}(v) + c$

$\displaystyle = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + c.$