$$\int u^3 \frac{\sec^2 x}{\sec^2 x}\, du = \frac{1}{4}u^4 + c = \frac{1}{4}\tan^4 + c.$$

- a even, b odd : write as powers of sec(x) and use integration by parts.

Example $\int \sin(3x)\cos(2x)\,dx$ useful fact: $\sin(A+B) = \sin A\cos B + \cos A \sin B$ ①
$\sin(A-B) = \sin A\cos B - \cos A\sin B$ ②

① + ② : $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$$\int \sin(3x)\cos(2x)\,dx = \frac{1}{2}\int \sin(5x) + \sin(x)\,dx = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c.$$

Example $\int \cos(4x)\cos(7x)\,dx$ useful fact: $\cos(A+B) = \cos A\cos B - \sin A\sin B$ ①
$\cos(A-B) = \cos A\cos B + \sin A\sin B$ ②

① + ② : $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$$= \frac{1}{2}\int \cos(11x) + \cos(-3x)\,dx = \frac{1}{22}\sin(11x) + \frac{1}{6}\cos(3x) + c.$$

§7.3 Trig substitutions

aim: deal with $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$.
① ② ③

$\cos^2 x + \sin^2 x = 1 \iff \sin^2 x = 1 - \cos^2 x$ ①

$\iff 1 + \tan^2 x = \sec^2 x$ ②

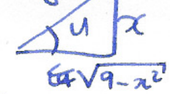
$\iff \cot^2 x + 1 = \operatorname{cosec}^2 x \iff \cot^2 x = \operatorname{cosec}^2 x - 1$ ③

Example $\int \sqrt{9-x^2}\,dx$ try $x = 3\sin u$
$\frac{dx}{du} = 3\cos u$

$$\int \sqrt{9 - 9\sin^2 u}\,\frac{dx}{du}\,du = \int 3\sqrt{1-\sin^2 u}\; 3\cos u\,du = \int 9\sqrt{\cos^2 u}\,\cos u\,du$$

$$= 9\int \cos^2 u\,du = \frac{9}{2}\int \cos 2u + 1\,du = \frac{9}{4}\sin 2u + \frac{9}{2}u + c$$

$$= \frac{9}{4} 2\sin u \cos u + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) = \frac{9}{2} x \sqrt{1-(x/3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + c$$



$\frac{x}{3} = \sin u$.

Example $\int \sqrt{1+4x^2}\,dx$ try: $x = \frac{1}{2}\tan u$

$$\frac{dx}{du} = \frac{1}{2}\sec^2 u$$

$$\int \sqrt{1+4\left(\tfrac{1}{2}\tan u\right)^2}\,\frac{dx}{du}\,du = \int \sqrt{1+\tan^2 u}\;\frac{1}{2}\sec^2 u\,du = \frac{1}{2}\int \sec^3 u\,du$$

$$= \frac{1}{2}\int \underbrace{\sec u}_{u}\,\underbrace{\sec^2 u}_{v'}\,du \qquad u = \sec u \quad v' = \sec^2 u$$

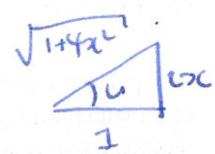
$$u' = \sec u \tan u \quad v = \tan u$$

$$= \frac{1}{2}\sec u \tan u - \frac{1}{2}\int \sec u \underbrace{\tan^2 u}_{\sec^2 u - 1}\,du$$

$$\frac{1}{2}\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\int \sec u\,du - \frac{1}{2}\int \sec^3 u\,du$$

$\tan u = 2x$

$$\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + c$$



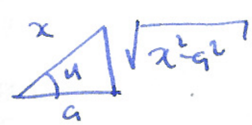
$$\frac{1}{2}\int \sec^3 u\,du = \frac{1}{4}\sec u \tan u + \frac{1}{4}\ln|\sec u + \tan u| + c$$

$$= \frac{1}{4}\frac{1}{\sqrt{1+4x^2}}2x + \frac{1}{4}\ln\left|\frac{1}{\sqrt{1+4x^2}} + 2x\right| + c$$

Example $\int \frac{1}{x^2\sqrt{x^2-a^2}}\,dx$ try $x = a\sec u$

$$\frac{dx}{du} = a\sec u \tan u$$

$$\int \frac{1}{a^2\sec^2 u}\,\frac{1}{\sqrt{a^2\sec^2 u - a^2}}\,\frac{dx}{du}\,du = \frac{1}{a^3}\int \frac{1}{\sec^2 u}\cdot\frac{1}{\tan u}\cdot a\sec u \tan u\,du$$



$$= \frac{1}{a^2}\int \cos u\,du = \frac{1}{a^2}\sin u + c = \frac{1}{a^2}\frac{\sqrt{a^2-x^2}}{a} + c = \frac{1}{a^2}\sqrt{1-(x/a)^2} + c$$

$\sec u = \frac{x}{a}$