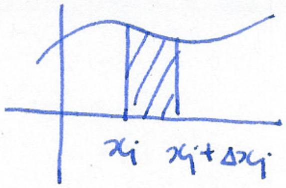
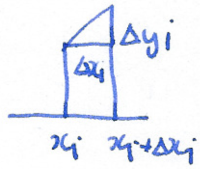
what goes wrong:



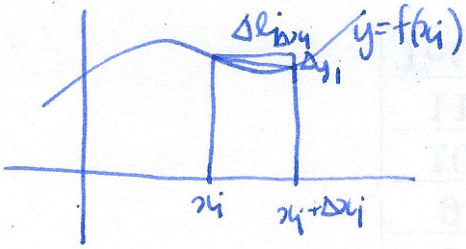
area of strip $\simeq y(x_i)\Delta x_i$ $\qquad \dfrac{\text{error}}{\text{area}} \to 0$ as $\Delta x_i \to 0$

surface area is like arc length



$\dfrac{\text{error}}{\cancel{\text{arc length}}\ \Delta x_i \text{ or } \Delta x_i+\Delta y_i} \not\to 0$ as $\Delta x_i \to 0$, in fact ratio is constant

right way:



arc length $\simeq \displaystyle\sum_{i=1}^{n} \Delta l_i \qquad \Delta l_i^2 = \Delta x_i^2 + \Delta y_i^2$

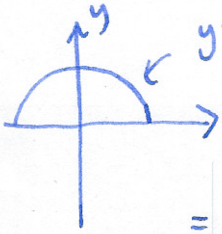
arc length $\simeq \displaystyle\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} \to \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

surface area of strip $\simeq 2\pi y_i \Delta l_i$

total surface area: $\displaystyle\sum_{i=1}^{n} 2\pi y_i \Delta l_i = \sum_{i=1}^{n} 2\pi\, y_i(x_i)\sqrt{(\Delta x_i)^2+(\Delta y_i)^2}$ $\leftarrow$ doesn't look like integral!

$$= 2\pi \sum_{i=1}^{n} y_i(x_i)\sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i \longrightarrow 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = S \text{ surface area.}$$

check: surface area of sphere $(4\pi r^2)$

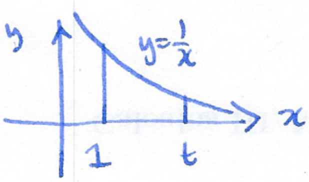


$y=\sqrt{1-x^2} \qquad \dfrac{dy}{dx} = \dfrac{1}{2}(1-x^2)^{-1/2}(-2x) \qquad$ surface area $= 2\pi\displaystyle\int_{-1}^{1}\sqrt{1-x^2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\pi\int_{-1}^{1}\sqrt{1-x^2}\sqrt{1+\frac{x}{1-x^2}}\,dx = 2\pi\int_{-1}^{1}\sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx = 2\pi\int_{-1}^{1} 1\,dx$$

$$= [2\pi x]_{-1}^{1} = 4\pi.$$

Example



between 1 and ∞

between 1 and t, let $t\to\infty$.

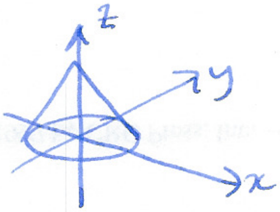
volume of revolution $V = \displaystyle\int_1^t \frac{\pi}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^t = \pi\left(1-\frac{1}{t}\right) \to \pi$ as $t\to\infty$

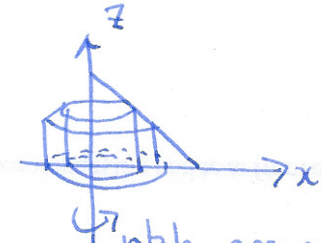
surface area: $A = 2\pi\displaystyle\int_1^t \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi\int_1^t \frac{1}{x}\,dx = 2\pi\left[\ln|x|\right]_1^t$

$= 2\pi\ln(t) \to \infty$ as $t\to\infty$.

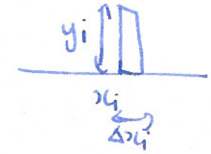
§6.4 Cylindrical shells

volume of cone:





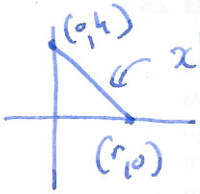
rotate around y-axis.



volume of shell $V_i \approx 2\pi x_i y_i \Delta x_i$ total volume $V \approx \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} 2\pi x_i y_i \Delta x_i$

$$V = 2\pi \int_a^b x f(x)\, dx.$$

Example



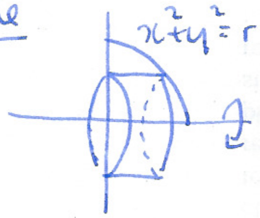
$xh + yr = hr$

$$V = \int_0^r 2\pi x \left(\frac{hr - xh}{hr} \right) dx$$

$$V = \frac{2\pi h}{r} \int_0^r x(r-x)\, dx = 2\pi \frac{h}{r} \int_0^r xr - x^2\, dx$$

$$= 2\pi \frac{h}{r} \left[\frac{1}{2} x^2 r - \frac{1}{3} x^3 \right]_0^r = 2\pi \frac{h}{r} \left(\frac{1}{2} r^3 - \frac{1}{3} r^3 \right) = \frac{1}{3} \pi r^2$$

Example



$x^2 + y^2 = r^2$ volume of hemisphere: rotate about x-axis.

$$V = \int 2\pi y\, f(y)\, dy$$

$$V = \int_0^r 2\pi y \sqrt{r^2 - y^2}\, dy = 2\pi \left[\frac{2}{3} (r^2 - y^2)^{3/2} \cdot -\frac{1}{2} \right]_0^r = -\frac{2\pi}{3} (0 - r^3) = \frac{2}{3} \pi r^3.$$

§7.1 Integration by parts "reverse product rule"

product rule: $(uv)' = u'v + uv'$

$$\int (uv)'\, dx = \int u'v\, dx + \int uv'\, dx$$

$$uv = \int u'v\, dx + \int uv'\, dx$$

$$\boxed{\int uv'\, dx = uv - \int u'v\, dx}$$ ← integration by parts formula.

Example $\int x \sin(x)\, dx$

$u = x$ $\quad v' = \sin(x)$

$u' = 1$ $\quad v = -\cos(x)$

$$\int \underbrace{x}_{u} \underbrace{\sin x}_{v'}\, dx = \underbrace{x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(1)}_{u'} \underbrace{(-\cos x)}_{v}\, dx$$