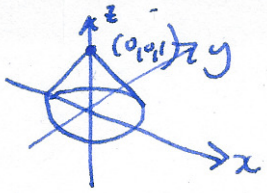
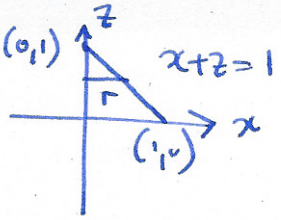
Example find volume of cone



$A = \pi r^2$

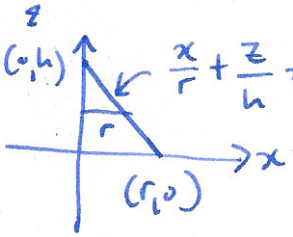


$r = x$

$$V = \int_0^1 A(z)\,dz = \int_0^1 \pi r^2\,dz = \int_0^1 \pi x^2\,dz = \int_0^1 \pi (1-z)^2\,dz = \int_0^1 \pi (1-2z+z^2)\,dz$$

$$= \pi \left[z - z^2 + \frac{1}{3}z^3 \right]_0^1 = \pi \left(1 - 1 + \frac{1}{3}\right) = \frac{\pi}{3}.$$

in general

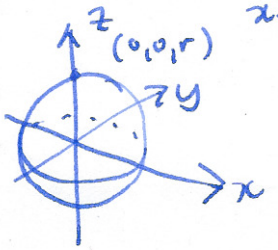


$\frac{x}{r} + \frac{z}{h} = 1 \qquad x = r\left(1 - \frac{z}{h}\right)$

$$V = \int_0^h A(z)\,dz = \int_0^h \pi r^2\,dz = \int_0^h \pi r^2 \left(1 - \frac{z}{h}\right)^2 dz = \pi r^2 \int_0^h 1 - \frac{2z}{h} - \frac{z^2}{h^2}\,dz$$

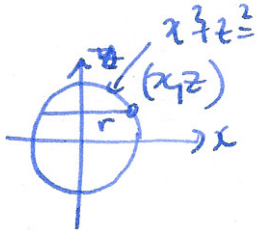
$$= \pi r^2 \left[z - \frac{z^2}{h} + \frac{z^3}{3h^2} \right]_0^h = \pi r^2 \left(h - h + \frac{h}{3}\right) = \frac{1}{3}\pi r^2 h$$

Example volume of a sphere



$x^2 + y^2 + z^2 = r^2$

$$V = \int_{-r}^{r} A(z)\,dz$$

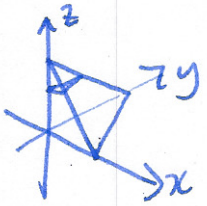


$x = \sqrt{r^2 - z^2}$

$$= \int_{-r}^{r} \pi (r^2 - z^2)\,dz = \pi \left[r^2 z - \frac{1}{3}z^3 \right]_{-r}^{r} = 2\pi \left(r^3 - \frac{1}{3}r^3\right)$$

$$= \frac{4}{3}\pi r^3$$

Example tetrahedron with vertices $(0,0,0), (0,0,1), (0,1,0), (1,0,0)$.



$$V = \int_0^1 A(z)\,dz$$

area of triangle $= \frac{1}{2}$ base $\times$ height

$$= \frac{1}{2}(1-z)(1-z)$$

$$= \int_0^1 \frac{1}{2}(1-z)^2\,dz = \cdots = \frac{1}{6}.$$

Average values

recall: average value of list of numbers $a_1, \ldots, a_n$ is $\dfrac{a_1 + a_2 + \cdots + a_n}{n} = \dfrac{1}{n}\sum_{i=1}^{n} a_i$

Q: what about average value of function $f(x)$ on an interval $[a,b]$?