general rules

f	f'
$kf(x)$	$kf'(x)$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$
$f(g(x))$	$f'(g(x))\cdot g'(x)$
$\frac{f(x)}{g(x)}$	$\frac{g(x)f'(x)-g'(x)f(x)}{g(x)^2}$

f	$\int f(x)\,dx$
$kf(x)$	$k\int f(x)\,dx$
$f(x)+g(x)$	$\int f(x)\,dx+\int g(x)\,dx$.

§5.7 Integration by substitution

$$\int_{x=a}^{x=b} f(x)\,dx,\ x(u) \rightsquigarrow \int_{u=x^{-1}(a)}^{u=x^{-1}(b)} f(x(u))\,\frac{dx}{du}\,du$$

recall $\frac{dx}{du} = \frac{1}{\frac{du}{dx}}$ integration by substitution is "reverse chain rule".

Examples · $\int x\sin(x^2)\,dx$ try $x^2=u$ $\frac{du}{dx}=2x$

$$\cdot \int x\sin(x^2)\frac{dx}{du}\,du = \int x\sin(u)\frac{1}{2x}\,du = \int \frac{1}{2}\sin(u)\,du$$

$$= -\frac{1}{2}\cos(u)+c = -\cos(x^2)+c.$$ check!

$\cdot \int e^{4x}\,dx$ $u=4x$, $\frac{du}{dx}=4$ $\int e^u\frac{1}{4}\,du = \frac{1}{4}e^u+c = \frac{1}{4}e^{4x}+c$ check!

$\cdot \int \tan\theta\,d\theta = \int\frac{\sin\theta}{\cos\theta}\,d\theta$ [hint: $\frac{d}{dx}(\ln(f(x)) = \frac{1}{f(x)}\cdot f'(x)$]

try: $u=\cos\theta$, $\frac{du}{d\theta}=-\sin\theta$ $\int\frac{\sin\theta}{u}\cdot\frac{-1}{\sin\theta}\,d\theta = -\int\frac{1}{u}\,du = -\ln|u|+c$

$$= -\ln|\cos x|+c = \ln|\sec x|+c.$$ check!

- $\int x\sqrt{x+1}\,dx$ try $u=x+1$, $\frac{du}{dx}=1$ $\quad \int x\sqrt{u}\,du = \int (u-1)\sqrt{u}\,du$

$= \int u^{3/2}-u^{1/2}\,du = \frac{2}{5}u^{5/2}-\frac{2u^{3/2}}{3}+C = \frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c$

Definite integrals

$\int_0^2 x^2\sqrt{x^3+1}\,dx$ $\quad u=x^3+1$, $\frac{du}{dx}=3x^2$ $\quad \int_1^9 x^2\sqrt{u}\,\frac{1}{3x^2}\,du = \int_1^9 \frac{1}{3}u^{1/2}\,du = \left[\frac{2}{9}u^{3/2}\right]_1^9$

$= \frac{2}{9}27-\frac{2}{9}.$

§5.8 More functions

inverse trig functions

$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x)+c$

$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$ $\qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$

$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{|x|\sqrt{x^2-1}}$ $\qquad \int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1}(x)+c$

Examples $\int_0^1 \frac{1}{4+x^2}\,dx$ try $x=2u$, $x^2=4u^2$ $\quad \frac{dx}{du}=2 \rightsquigarrow \int_0^{1/2}\frac{1}{4+4u^2}\,2\,du$

$= \frac{1}{2}\int_0^{1/2}\frac{1}{1+u^2}\,du = \left[\frac{1}{2}\tan^{-1}(u)\right]_0^{1/2} = \frac{1}{2}\left(\tan^{-1}(1/2)-\tan^{-1}(0)\right).$

Exponential functions

$\frac{d}{dx}(e^x)=e^x$ $\qquad \int e^x\,dx = e^x+c$

$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{x\ln(b)}\right) = \ln(b)\,e^{x\ln(b)} = \ln(b)\,b^x$

$\int b^x\,dx = \frac{1}{\ln(b)}b^x+c$ $\qquad$ Examples $\int_0^3 10^x\,dx$ $\qquad \int_0^{\pi/2}\cos\theta\,10^{\sin\theta}\,d\theta$