Math 232 Calculus 2 Spring 20 Midterm 2a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use a calculator, and a US letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find $\int \cos 7x \cos 3x \, dx$.

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$.

$\int \cos 7x \cos 3x \, dx = \frac{1}{2} \int \cos 10x + \cos 4x \, dx$

$= \frac{1}{20} \sin 10x + \frac{1}{8} \sin 4x + C$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

(2) (10 points) Find $\int \tan^3 x \, dx$.

$= \int \tan^2 x \cdot \tan x \, dx = \int (\sec^2 x - 1) \tan x \, dx$

$= \int \sec^2 x \tan x - \tan x \, dx$ ①

$\int \tan x \, dx = \ln |\sec x| + c$

$\int \underset{v'}{\sec^2 x} \, \underset{u}{\tan x} \, dx$

$\int uv' \, dx = uv - \int u'v \, dx$

$u = \tan x \qquad v' = \sec^2 x$

$u' = \sec^2 x \qquad v = \tan x$

$\int \sec^2 x \tan x \, dx = \tan^2 x - \int \sec^2 x \tan x \, dx$

$\int \sec^2 x \tan x \, dx = \frac{1}{2} \tan^2 x$

10 ① $= \frac{1}{2} \tan^2 x - \ln |\sec x| + c$

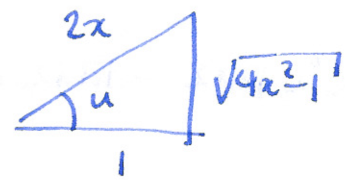
$\sin^2 u + \cos^2 u = 1$

$\tan^2 u + 1 = \sec^2 u$

(3) (10 points) Find $\displaystyle\int \frac{1}{x^2\sqrt{4x^2-1}}\,dx.$

$2x = \cancel{\tan u} \cdot \sec u.$

$2\frac{dx}{du} = \sec u \tan u$



$$\int \frac{1}{\frac{1}{4}\cancel{\tan}^2_{\sec} u\,\sqrt{\cancel{\tan}^2_{\sec} u-1}}\,\frac{dx}{du}\,du$$

$$\int \frac{4}{\sec^2 u \cdot \tan u}\,\frac{1}{2}\sec u \tan u\,du = 2\int \cos u\,du = +2\sin u + C$$

$$= \frac{2\sqrt{4x^2-1}}{2x} + C = \frac{\sqrt{4x^2-1}}{x} + C.$$

(4) (10 points) Find $\displaystyle\int \frac{x^2+x}{(x-1)(x^2+1)}\,dx$.

$$\frac{x^2+x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$x^2+x = A(x^2+1) + (Bx+C)(x-1)$$

$x=1$: $\quad 2 = 2A \quad \Rightarrow \quad A=1$

$$x^2+x = x^2+1 + Bx^2 + x(C-B) - C.$$

$$x-1 = Bx^2 + x(C-B) - C \qquad B=0,\ C=1.$$

$$\int \frac{1}{x-1} + \frac{1}{x^2+1}\,dx = \ln|x-1| + \tan^{-1}(x) + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(5) (10 points) Find $\int_0^\infty xe^{-2x}\,dx$.

$$\lim_{R\to\infty}\int_0^R \underbrace{x}_{u}\,\underbrace{e^{-2x}}_{v'}\,dx = \lim_{R\to\infty} \underbrace{\left[x\cdot -\tfrac{1}{2}e^{-2x}\right]_0^R}_{\to 0} - \int_0^R -\tfrac{1}{2}e^{-2x}\,dx$$

$u = x \qquad v' = e^{-2x}$

$u' = 1 \qquad v = -\frac{1}{2}e^{-2x}$

$$= \lim_{R\to\infty}\int_0^R \frac{1}{2}e^{-2x}\,dx = \lim_{R\to\infty}\left[-\frac{1}{4}e^{-2x}\right]_0^R$$

$$= \lim_{R\to\infty} \underbrace{-\frac{1}{4}e^{-2R}}_{\to 0 \text{ as } R\to\infty} + \frac{1}{4} = \frac{1}{4}$$

(6) (10 points) Find the degree three Taylor polynomial for $f(x) = e^{\sin x}$ centered at $x = 0$.

$f(x) = e^{\sin x}$

$f'(x) = e^{\sin x} \cdot \cos x$

$f''(x) = e^{\sin x} \cdot \cos^2 x + e^{\sin x} \cdot -\sin x$

$f^{(3)}(x) = e^{\sin x} \cdot \cos^3 x + e^{\sin x} \cdot 2\cos x \cdot -\sin x + e^{\sin x} \cdot -\sin^2 x + e^{\sin x} \cdot -\cos x$.

$f(0) = 1$

$f'(0) = 1$

$f''(0) = 1$

$f^{(3)}(0) = \cancel{4}\,0$

$T_3(x) = 1 + x + \frac{x^2}{2!} + \cancel{\frac{2x^3}{3!}}$

(7) Does the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge or diverge? If it converges, find the exact value.

$n=1: \quad 1 = 2B.$

$n=-1: \quad 1 = -2A.$

$$\frac{1}{n^2-1} = \frac{A}{n+1} + \frac{B}{n-1} = \frac{A(n-1)+B(n+1)}{n^2-1}.$$

$$= \frac{1/2}{n-1} + \frac{-1/2}{n+1}.$$

$$S_N = \frac{1/2}{1} - \frac{1/2}{3} + \frac{1/2}{2} - \frac{1/2}{4} + \frac{1/2}{3} - \frac{1/2}{5} + \frac{1/2}{4} - \frac{1/2}{6} + \cdots$$

etc ...

$$S_N = \frac{1}{2} + \frac{1}{4} - \frac{1/2}{N} - \frac{1/2}{N+1}$$

$$\lim_{N\to\infty} \frac{3}{4} - \frac{1/2}{N} - \frac{1/2}{N+1} = \frac{3}{4}$$

(8) Does the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converge or diverge?

limit comparison test with $b_n = \frac{1}{n}$ (both positive series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+1/n^2} = 1$$

$0 < 1 < \infty \Rightarrow$ as $\sum \frac{1}{n}$ diverges (p-series w/ $p=1$).

this implies $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

(9) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2-1}$ converge or diverge?

Limit comparison test with $b_n = \dfrac{1}{n^{3/2}}$ (both positive series)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2-1}\, n^{3/2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-1/n^2} = 1$$

$0 < 1 < \infty \Rightarrow$ as $\sum \frac{1}{n^{3/2}}$ converges (p-series w/ $p>1$)

then $\sum \frac{\sqrt{n}}{n^2-1}$ converges.

(10) Find the power series for $f(x) = x\sin(2x)$. What is the radius of convergence?

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots$$

$$x\sin(2x) = 2x^2 - \frac{2^3x^4}{3!} + \frac{2^5x^6}{5!} - \frac{(2x)^8}{7!} + \cdots$$

n-th non-zero term is $\dfrac{2^{2n-1}x^{2n}}{(2n-1)!}$

ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{2n+1}x^{2n+2}}{(2n+1)!} \frac{(2n-1)!}{2^{2n-1}x^{2n}} \right|$.

$= \lim_{n\to\infty} \frac{4|x^2|}{(2n)(2n+1)} = 0$ so converges for all x ($R = \infty$).