Sample midterm 2 Solutions

Q1 $\int \cos^3 x\,dx = \int \cos^2 x \cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$

$u = \sin x \quad \frac{du}{dx} = \cos x \qquad = \int (1-u^2)\cos x \frac{dx}{du}\,du$

$= \int (1-u^2)\cos x \frac{1}{\cos x}\,du = \int 1-u^2\,du = u - \frac{1}{3}u^3 + c$

$= \sin x - \frac{1}{3}\sin^3 x + c$

Q2 $\int \cos \underbrace{6x}_{B} \sin \underbrace{5x}_{A}\,dx$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B.$

$\sin(A+B) + \sin(A-B) = 2\sin A\cos B.$

$= \frac{1}{2}\int \sin 11x + \sin(-x)\,dx$

$= -\frac{1}{22}\cos x + \frac{1}{2}\cos x + c$

Q3 $\int \frac{x}{\sqrt{16x^2+1}}\,dx \qquad u = 16x^2+1 \quad \frac{du}{dx} = 32x$

$\int \frac{x}{\sqrt{u}} \cdot \frac{dx}{du}\,du = \int x\,u^{-1/2}\frac{1}{32x}\,du$

$= \frac{1}{32}\int u^{-1/2}\,du = \frac{1}{32}\left[2u^{1/2}\right. + c = \frac{1}{16}\sqrt{16x^2+1} + c$

Q4 $\displaystyle\int \frac{x^2-5x-2}{(x-1)^2(x+3)}\,dx$

$$\frac{x^2-5x-2}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}.$$

$$x^2-5x-2 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

$x=1$: $-6 = 4B$ $\qquad B = -3/2$

$x=-3$: $22 = 16C$ $\qquad C = 11/8$ $\qquad A = \frac{31}{12}$

$x=0$: $-2 = -3A + 3B + C$ $\qquad -3A = -2 - 3B - C$

$$A = \frac{1}{3}\left(2 + \frac{9}{2} + \frac{11}{8}\right) = \frac{1}{3}\,\frac{16+36+11}{8} = \frac{62}{3\cdot 8}$$

$$\int \frac{\frac{31}{12}}{x-1} - \frac{3/2}{(x-1)^2} + \frac{11/8}{x+3}\,dx$$

$$= \frac{31}{12}\ln|x-1| + \frac{3}{2}(x-1)^{-1} + \frac{11}{8}\ln|x+3| + c$$

Q5 $\displaystyle\int_0^1 x^2 \ln x^4\,dx = \int_0^1 \underbrace{4x^2}_{v'}\,\underbrace{\ln x}_{u}\,dx$

$\int uv'\,dx = uv - \int u'v\,dx$

$u = \ln x \qquad v' = 4x^2$

$u' = \frac{1}{x} \qquad v = \frac{4}{3}x^3.$

$$= \left[\ln x \cdot \frac{4}{3}x^3\right] - \int \frac{4}{3}x^2\,dx$$

$$= \frac{4}{3}x^3\ln x - \frac{4}{9}x^3 + c$$

$$\lim_{R\to 0}\left[\frac{4}{3}x^3\ln x-\frac{4}{9}x^3\right]_R^1 = \lim_{R\to 0} -\frac{4}{9}-\frac{4}{3}R^3\ln R+\underbrace{\frac{4}{9}R^3}_{\to 0} \quad \circledast$$

$$\lim_{R\to 0} R^3\ln R = \lim_{R\to 0}\frac{\ln R}{R^{-3}} \overset{\text{l'Hôpital}}{=} \lim_{R\to 0}\frac{1/R}{-3R^{-4}} = \lim_{R\to 0} -\frac{1}{3}R^3 = 0$$

so $\circledast = -\frac{4}{9}$.

Q6 $\int_0^\infty \frac{1}{x^2+9}\,dx \quad \circledast$ use: $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)+c$

$3u = x \qquad \frac{du}{dx}=\frac{1}{3} \qquad \int\frac{1}{x^2+9}\,dx = \int\frac{1}{9u^2+9}\cdot\frac{dx}{du}\cdot du$

$$= \int\frac{1}{9}\,\frac{1}{1+u^2}\cdot 3\,du = \frac{1}{3}\int\frac{1}{1+u^2}\,du = \frac{1}{3}\tan^{-1}(u)+c = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

$$\circledast = \lim_{R\to\infty}\cancel{\int_0^R}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\cancel{dx}\right]_0^R = \lim_{R\to\infty}\frac{1}{3}\tan^{-1}\left(\frac{R}{3}\right)-0 = \frac{\pi}{6}.$$

Q7 $f(x) = x^{3/2}$

$f'(x) = \frac{3}{2}x^{1/2}$

$f''(x) = \frac{3}{4}x^{-1/2} \leftarrow$ not defined at $x=0$, so no deg 3 Taylor polynomial

$f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$

$$T_3(x) = 1+\frac{3}{2}(x-1)+\frac{3}{4}\frac{(x-1)^2}{2!}+-\frac{3}{8}\frac{(x-1)^3}{3!}$$

centered at 1

find error bound for $2^{3/2}$ $n=3$. $|c-x| = (1-2) = 1$. ④

$$\left|f(x) - T_3(x)\right| \leq K\frac{(c-x)^{n+1}}{(n+1)!}$$ K upper bound for $|f^{(4)}(x)|$ on $[1,2]$

$f^{(4)}(x) = -\frac{9}{16}x^{-5/2}$ so $|f^{(4)}(x)| \leq \frac{9}{16}$ on $[1,2]$.

so $\left|2^{3/2} - T_3(2)\right| \leq \frac{9}{16}\cdot\frac{1}{4!} = \frac{3}{128}$

Q8

$$a_n = \frac{2^n}{n!} = \underbrace{\frac{2}{1}}_{2}\cdot\underbrace{\frac{2}{2}}_{\leq 1}\cdot\underbrace{\frac{2}{3}}_{\leq 1}\cdots\underbrace{\frac{2}{n-1}}_{\leq 1}\cdot\frac{2}{n} \leq 2\cdot\frac{2}{n} \to 0 \text{ as } n\to\infty$$

~~as~~ so $a_n \to 0$ as $n\to\infty$.

Q9

$$\sum_{n=2}^{\infty} e^{-n} = e^{-2} + e^{-3} + \cdots = \frac{e^{-2}}{1-e}$$ converges.

Q10

$$\frac{1}{4n^2+8n+3} = \frac{1}{(2n+1)(2n+3)} = \frac{A}{2n+1} + \frac{B}{2n+3}$$

$1 = A(2n+3) + B(2n+1)$ $n=-\frac{1}{2}$: $1 = 2A$ $A = 1/2$

$n=-\frac{3}{2}$: $1 = -2B$. $B = -1/2$.

$$\sum_{n=1}^{\infty} \frac{1/2}{2n+1} - \frac{1/2}{2n+3}.$$ $s_1 = \frac{1/2}{3} - \frac{1/2}{5}$

$s_2 = \frac{1/2}{3} - \frac{1/2}{5} + \frac{1/2}{5} - \frac{1/2}{7}$

$s_N = \frac{1/2}{3} - \frac{1/2}{2N+3}$ so $\lim_{N\to\infty} s_N = 1/6$.

Q11 $\cos\left(\frac{1}{n}\right) \to 1$ as $n \to \infty$ so $\sum \cos\left(\frac{1}{n}\right)$ diverges.

Q12 comparison test. $\ln(n) < n$ for $n \geq 1$ so $(\ln(n))^2 \leq n^2$

so $\frac{(\ln(n))^2}{n^4} \leq \frac{n^2}{n^4} = \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges (p-series with $p>1$)

so $\sum \frac{(\ln(n))^2}{n^4}$ converges.

Q13 $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n}\right|$

$= \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1 \Rightarrow$ converges.

Q14 $\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}$ (*) consider $\left|\frac{n \sin n}{n^3+1}\right|$ $|\sin n| \leq 1$.

so $\left|\frac{n \sin n}{n^3+1}\right| \leq \frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}$ $\sum \frac{1}{n^2}$ converges, p-series w $p>1$.

so $\sum \left|\frac{n \sin n}{n^3+1}\right|$ converges by comparison test, so (*) absolutely convergent, so converges.

Q15 $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ limit comparison test with $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{n^2}{n^3+1} \cdot n = \lim_{n\to\infty} \frac{n^3}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+1/n^3} = 1 < \infty$ so $0 <$

as $\sum \frac{1}{n}$ diverges, $\sum \frac{n^2}{n^3+1}$ diverges.

Q16 $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ ratio test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right|$ 6

$= \lim_{n\to\infty} |x|\left(\frac{1}{1+1/n}\right)^2 = |x|$ so converges for $|x| < 1$.

check endpoints: $x = 1$ converges as p-series w/ $p > 1$

$x = -1$ converges as alternating series test works so

converges for $|x| \leq 1$.

Q17 $f(x) = \cos(x^{1/2})$

$f'(x) = -\sin(x^{1/2}) \cdot \frac{1}{2}x^{-1/2}$

$f''(x) = -\cos(x^{1/2}) \cdot \frac{1}{4x} + -\sin(x^{1/2}) \cdot -\frac{1}{4}x^{-3/2}$

$f^{(3)}(x) = \sin(x^{1/2}) \cdot \frac{1}{8x^{3/2}} + -2\cos(x^{1/2}) \cdot -\frac{1}{4x^2} - \cos(x^{1/2})\frac{3}{8}x^{-5/2}$.

$f(1) = \cos(1)$

$f'(1) = -\frac{1}{2}\sin(1)$

$f''(1) = -\frac{1}{4}\cos(1) + \frac{1}{4}\sin(1)$

$f'''(1) = \frac{1}{8}\sin(1) + \frac{1}{2}\cos(1) - \frac{3}{8}\cos(1)$

$T_3(x) = \cos(1) + \left(-\frac{1}{2}\sin(1)\right)(x-1)$

$+ \left(\frac{1}{4}\sin(1) - \frac{1}{4}\cos(1)\right)(x-1)^2$

$+ \left(\frac{1}{8}\sin(1) + \frac{1}{8}\cos(1)\right)(x-1)^3$.

Q18 $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^8}{3!} + \cdots$$

$$x^2 e^{-x^2} = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^{10}}{3!} + \cdots$$

Q19. $\lim_{x \to 0} e^{-1/x} = \lim_{y \to \infty} e^{-y} = 0 \quad (\text{set } y = \frac{1}{x})$.

$$\lim_{x\to 0} \frac{e^{-1/x}}{x^n} = \lim_{y\to\infty} \frac{e^{-y}}{(1/y)^n} = \lim_{y\to\infty} y^n e^{-y} = \lim_{y\to\infty} \frac{y^n}{e^y} \overset{\text{l'Hôpital}}{=} \lim_{y\to\infty} \frac{ny^{n-1}}{e^y}$$

$$\overset{\text{l'Hôpital}}{=} \cdots = \lim_{y\to\infty} \frac{n!}{e^y} = 0.$$

$$f(x) = e^{-1/x}$$

$$f'(x) = e^{-1/x} \cdot -x^{-2} = \frac{e^{-1/x}}{x^2}.$$

$$f''(x) = e^{-1/x} \cdot x^{-4} + e^{-1/x} \cdot 2x^{-3} = \frac{e^{-1/x}}{x^4}(1+2x)$$

set $p_1(x) = 1$ ✓ base case of induction

$$\left(\frac{p_n(x)}{x^{2n}} \cdot e^{-1/x}\right)' = \frac{x^{2n} \cdot p' - p\, 2n x^{2n-1}}{x^{4n}} e^{-1/x} + \frac{p}{x^{2n}} e^{-1/x} \cdot + \frac{1}{x^2}.$$

$$= e^{-1/x}\left(\cancel{\frac{p' - 2nxp + x}{x^{2n}}} \; \frac{p'}{x^{2n}} - \frac{p2n}{x^{2n+1}} + \frac{p}{x^{2n+2}}\right)$$

$$= e^{-1/x}\left(\frac{x^2p' - p2nx + p}{x^{2(n+1)}}\right) \text{ so } p_{n+1}(x) = x^2p'(x) - p(x)2nx + p'(x)$$

as required.

Therefore as $\lim_{x\to 0} p_n(x) = p_n(0)$, $f^{(n)}(x)$ is cts for all n.