MTH 232 SMT1 Solutions

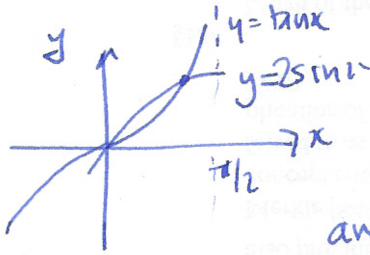
Q1 $\displaystyle\int \frac{\cos x}{1-\sin x}\,dx$ sub $u = 1-\sin x$, $\dfrac{du}{dx} = -\cos x$

$$\int \frac{\cos x}{u}\cdot\frac{dx}{du}\,du = \int \frac{\cos x}{u}\cdot\frac{1}{-\cos x}\,dx$$

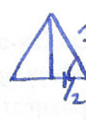
$$= \int -\frac{1}{u}\,du = -\ln|u| + c = -\ln|1-\sin x| + c.$$

Q2 $$\int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{1}{\cos x}\,dx = \int \sec x\,dx = \ln|\sec x + \tan x| + c$$

Q3



intersection point: $\tan x = 2\sin x$ $\quad \dfrac{\sin x}{\cos x} = 2\sin x \quad \cos x = \dfrac{1}{2}.$

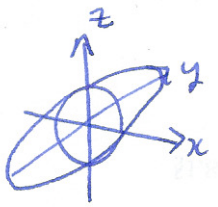


$x = \pi/3 > 1$ not in $[0,1]$.

$$\text{area} = \int_0^1 2\sin x - \tan x\,dx = \Big[-2\cos x + \ln|\sec x|\Big]_0^1.$$

$$= -2\cos(1) + \ln|\sec(1)| + 2$$

Q4



$\perp$ to y-axis: $V = \displaystyle\int_{-4}^{4} A(y)\,dy$ cross-section: $4x^2 + 4y^2 = 16 - y^2.$

circle of radius $\sqrt{4 - y^2/4}$.

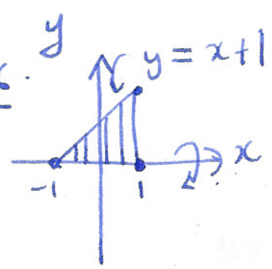
$A = \pi r^2 = \pi\,(4 - y^2/4)$.

$$V = \int_{-4}^{4} \pi\left(4 - \frac{y^2}{4}\right)dy = \pi\left[4y - \frac{y^3}{12}\right]_{-4}^{4} = \pi\left(16 - \frac{64}{12} + 16 - \frac{64}{12}\right)^{16}_{3}.$$

$$= \pi 16\left(2 - \frac{2}{3}\right) = \frac{64\pi}{3}$$

Q5 $$\frac{1}{3}\int_0^3 e^{-3x}\,dx = \frac{1}{3}\left[-\frac{1}{3}e^{-3x}\right]_0^3 = -\frac{1}{9}\left(e^{-9} - 1\right) = \frac{1}{9}\left(1 - e^{-9}\right).$$

Q6.

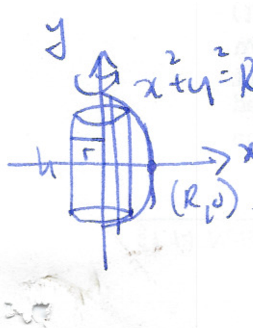


$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi r^2\,dx = \int_{-1}^{1} \pi y^2\,dx = \int_{-1}^{1} \pi (x+1)^2\,dx$$

$$= \pi\int_{-1}^{1} x^2 + 2x + 1\,dx = \pi\left[\frac{1}{3}x^3 + x^2 + x\right]_{-1}^{1} = \pi\left(\frac{1}{3} + 1 + 1 + \frac{1}{3} - 1 + 1\right)$$

$$= \frac{8\pi}{3}.$$

Q7



$$V = \int_0^R 2\pi r h\,dx = \int_0^R 2\pi x\, 2\sqrt{R^2 - x^2}\,dx \qquad u = R^2 - x^2 \quad \frac{du}{dx} = -2x.$$

$$= 4\pi\int_{R^2}^{0} x\sqrt{u}\,\frac{dx}{du}\,du = 2\pi\int_0^{R^2} u^{1/2}\,du = 2\pi\left[\frac{2u^{3/2}}{3}\right]_0^{R^2} = \frac{4\pi}{3}R^3.$$

Q8 $\int \underbrace{x^2}_{v'} \underbrace{\ln(x-2)}_{u} dx$ ⊛ $\int uv'\,dx = uv - \int u'v\,dx$ $u = \ln(x-2)$ $v' = x^2$ $u' = \frac{1}{x-2}$ $v = \frac{1}{3}x^3$.

$= \frac{1}{3}x^3\ln(x-2) - \int \frac{1}{x-2}\cdot\frac{1}{3}x^3\,dx$ sub $u = x-2$ $\frac{du}{dx} = 1$. $\int \frac{x^3}{3(x-2)}dx = \int \frac{(u+2)^3}{u}du$.

$= \int \frac{u^3+6u^2+12u+8}{u}du = \int u^2+6u+12+\frac{8}{u}du = \frac{1}{3}u^3+3u^2+12u+8\ln|u|+c$.

⊛ $= \frac{1}{3}x^2\ln(x-2) + \frac{1}{3}(x-2)^3 + 3(x-2)^2 + 12(x-2) + 8\ln(x-2) + c$

Q9. $\int \underbrace{e^{-3x}}_{u} \underbrace{\cos(2x)}_{v'} dx = uv - \int u'v\,dx = e^{-3x}\frac{1}{2}\sin(2x) - \int -3e^{-3x}\frac{1}{2}\sin 2x\,dx$.

$u = e^{-3x}$ $v' = \cos(2x)$ $u' = -3e^{-3x}$ $v = \frac{1}{2}\sin(2x)$ $u = e^{-3x}$ $v' = \sin 2x$ $u' = -3e^{-3x}$ $v = \frac{1}{2}\cos 2x$

$\int e^{-3x}\cos(2x)dx = \frac{1}{2}e^{-3x}\sin(2x) + \frac{3}{2}\int \underbrace{e^{-3x}}_{u}\underbrace{\sin 2x}_{v'}dx = \frac{1}{2}e^{-3x}\sin 2x + \frac{3}{2}\cdot\frac{1}{2}e^{-3x}\cos 2x - \frac{3}{2}\int -\frac{3}{2}e^{-3x}\cos 2x\,dx$

$-\frac{5}{4}\int e^{-3x}\cos 2x\,dx = \frac{1}{2}e^{-3x}\sin 2x + \frac{3}{4}e^{-3x}\cos 2x + c$

$\int e^{-3x}\cos(2x)dx = -\frac{2}{5}e^{-3x}\sin 2x - \frac{3}{5}e^{-3x}\cos 2x + c$

Q10 $\int \underbrace{x}_{u} \underbrace{e^{-x}\sin x}_{v'} dx$ $u = x$ $v' = e^{-x}\sin x$ $u' = 1$ $v = \int \underbrace{e^{-x}}_{w}\underbrace{\sin x}_{z'}dx$ $w = e^{-x}$ $z' = \sin x$ $w' = -e^{-x}$ $z = -\cos x$.

$\int e^{-x}\sin x\,dx = -e^{-x}\cos x + \int \underbrace{+e^{-x}}_{w}\underbrace{-\cos x}_{z'}dx = -e^{-x}\cos x + e^{-x}\sin x + \int -e^{-x}\sin x\,dx$. ⊛.

$w = e^{-x}$ $z' = \cos x$ $w' = -e^{-x}$ $z = \sin x$

$2\int e^{-x}\sin x\,dx = -e^{-x}\sin x + e^{-x}\cos x$ $\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}\sin x + \frac{1}{2}e^{-x}\cos x$.

$\int xe^{-x}\sin x\,dx = -\frac{1}{2}xe^{-x}(\sin x + \cos x) - \int -\frac{1}{2}\underbrace{e^{-x}\sin x}_{*} + \frac{1}{2}e^{-x}\cos x\,dx$. $**$

$**$ $\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int -e^{-x}\sin x\,dx = -e^{-x}\cos x + -e^{-x}\sin x - \int -e^{-x}\cos x\,dx$

$$\int e^{-x}\cos x\,dx = -\tfrac{1}{2}e^{-x}(\sin x+\cos x)+c.$$

$$\int x e^{-x}\sin x\,dx = -\tfrac{1}{2}xe^{-x}(\sin x-\cos x)+\tfrac{1}{2}\cdot -\tfrac{1}{2}e^{-x}(\sin x-\cos x)+\tfrac{1}{2}\cdot-\tfrac{1}{2}e^{-x}(\sin x+\cos x)+C$$

$$= -\tfrac{1}{2}xe^{-x}(\sin x-\cos x) - \frac{1}{\cancel{4}\,2}e^{-x}\sin x + C.$$

Q11 $\displaystyle\int_0^{\pi/2}\sin^2 x\cos^3 x\,dx$ $\quad u=\sin x$, $\quad \frac{du}{dx}=\cos x$ $\quad \displaystyle\int_0^{\pi/2} u^2\cos^2 x\cdot\cos x\,\frac{dx}{du}\,du$

$$= \int_0^{\pi/2} u^2\cos^2 x\,du = \int_0^{\pi/2} u^2(1-u^2)\,du = \int_0^{\pi/2} u^2-u^4\,du = \left[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right]_0^{\pi/2}$$

$$= \frac{1}{3}\frac{\pi^3}{8}-\frac{1}{5}\frac{\pi^5}{32}.$$

Q12 $\displaystyle\int \sin(7x)\cos(3x)\,dx$ $\quad \left.\begin{aligned}\sin(A+B)&=\sin A\cos B+\sin B\cos A\\ \sin(A-B)&=\sin A\cos B-\sin B\cos A\end{aligned}\right\}\ \sin(A+B)+\sin(A-B)=2\sin A\cos(B).$

$$= \frac{1}{2}\int\sin(10x)+\sin(4x)\,dx = -\frac{1}{20}\cos 10x-\frac{1}{8}\cos 4x + C.$$

Q13 $\displaystyle\int\frac{x^2}{\sqrt{x^2+4}}\,dx.$ $\quad x = 2\tan u.$ $\quad \frac{dx}{du}=2\sec^2 u.$ $\quad \displaystyle\int\frac{4\tan^2 u}{\sqrt{4\tan^2 u+4}}\cdot\frac{dx}{du}\,du = \int\frac{4\tan^2 u}{2\sec u}\cdot 2\sec^2 u\,du.$

$$= 4\int\frac{\sin^2 u}{\cos^2 u}\cdot\frac{1}{\cos u}\,du = 4\int\tan^2 u\sec u\,du. = 4\int\underset{a}{\tan u}\,(\underset{b'}{\tan u\sec u})\,du$$

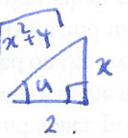
$a=\tan u \quad b'=\tan u\sec u$

$a'=\sec^2 u \quad b=\sec u$

$$4\int\tan^2 u\sec u\,du = 4\tan u\sec u - 4\int\sec^2 u\sec u\,du = 4\tan u\sec u - 4\int(\tan^2 u+1)\sec u\,du$$

$$8\int\tan^2 u\sec u\,du = 4\tan u\sec u - 4\int\sec u\,du = 4\tan u\sec u - 4\ln|\sec u+\tan u| + C.$$

$$4\int\tan^2 u\sec u\,du = 2\tan u\sec u - 2\ln|\sec u+\tan u| + C.$$



$\tan u = x/2$

$\sec u = \frac{1}{\cos u} = \frac{1}{2}\sqrt{x^2+4}$

$$= 2\cdot\frac{x}{2}\cdot\frac{1}{2}\sqrt{x^2+4} - 2\ln\left|\frac{1}{2}\sqrt{x^2+4}+\frac{x}{2}\right| + C.$$

Q14 $\displaystyle\int\sqrt{4x^2-1}\,dx$ $\quad x=\frac{1}{2}\sec u.$ $\quad \frac{dx}{du}=\frac{1}{2}\sec u\tan u.$ $\quad \displaystyle\int\sqrt{4\cdot\tfrac{1}{4}\sec^2 u-1}\,\frac{dx}{du}\,du$

$= \int \tan u \cdot \frac{1}{2} \sec u \tan u \, du = \frac{1}{2} \int \tan^2 u \sec u \, du$ see prev Q.

<u>Q15</u> $\int \frac{x}{\sqrt{1-2x^2}} dx$ $u = 1-2x^2$, $\frac{du}{dx} = -4x$ $\int \frac{x}{\sqrt{u}} \cdot \frac{dx}{du} du = \int \frac{x}{\sqrt{u}} \cdot \frac{1}{-4x} du$

$= -\frac{1}{4} \int u^{-1/2} du = -\frac{1}{4} 2 u^{1/2} + c = -\frac{1}{2}\sqrt{1-2x^2} + c.$