Math 232 Calculus 2 Spring 20 Final b

Name: Solutions

- I will count your best 10 of the following 12 questions.
- You may use a calculator without Computer Algebra System capabilities, and a single US letter page of notes, but no phones or other assistance.
- You must show your work to receive credit for a question.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
	100	

Final	
Overall	

$$\int uv'\,dx = uv - \int u'v\,dx$$

(1) Find $\int \underbrace{x^2}_{u}\underbrace{e^{-3x}}_{v'}\,dx$.

$$= x^2 \cdot -\frac{1}{3}e^{-3x} - \int 2x \cdot -\frac{1}{3}e^{-3x}\,dx$$

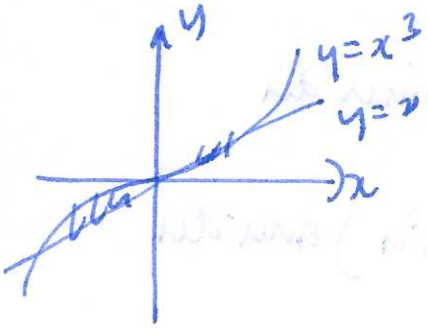
$$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int \underbrace{x}_{u}\underbrace{e^{-3x}}_{v'}\,dx$$

$$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left(x \cdot -\frac{1}{3}e^{-3x} - \int -\frac{1}{3}e^{-3x}\,dx\right).$$

$$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\int e^{-3x}\,dx$$

$$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x} + c$$

(2) Sketch the region bounded by the curves $y = x$, and $y = x^3$, and find the area of this region.



intersections $x = x^3$.

$x(x^2-1)=0$ $\quad x(x-1)(x+1)=0$

$$\int_{-1}^{0} x^3 - x\,dx + \int_{0}^{1} x - x^3\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^{0} + \left[-\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_{0}^{1} = -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} = \frac{1}{2}$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

(3) Find $\int \sin^4(x)\,dx$. $= \int \underset{v'}{\sin x}\;\underset{u}{\sin^3 x}\,dx$

$$\int \sin^4 x\,dx = \sin^3 x \cdot -\cos x - \int 3\sin^2 x\cos x \cdot -\cos x\,dx$$

$$\int \sin^4 x\,dx = -\sin^3 x\cos x + 3\int \sin^2 x \underset{1-\sin^2 x}{\underset{\parallel}{\cos^2 x}}\,dx.$$

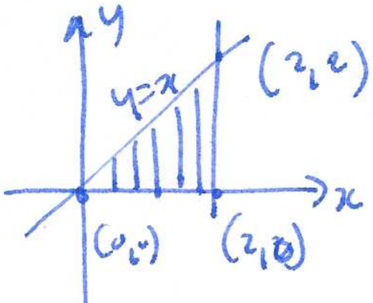
$$4\int \sin^4 x\,dx = -\sin^3 x\cos x + \underbrace{3\int \sin^2 x\,dx}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$= 3\int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx = \frac{3}{2}x - \frac{3}{4}\sin 2x + c$$

$$\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x - \frac{3}{16}\sin 2x + c$$

(4) Find the volume of revolution obtained by rotating the triangle determined by the points $(0,0)$, $(2,2)$ and $(2,0)$ around the x-axis.



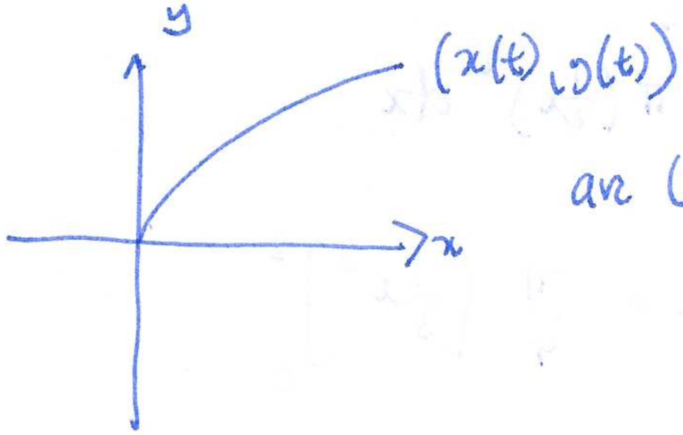
discs

$$\int_0^2 \pi (x)^2 \, dx$$

$$= \left[\pi \frac{1}{3} x^3 \right]_0^2 = \frac{8\pi}{3}.$$

$$\frac{dx}{dt} = t^2 \quad \frac{dy}{dt} = t.$$

(5) Sketch the parameterized curve given by $x(t) = \frac{1}{3}t^3$ and $y(t) = \frac{1}{2}t^2$. Use the parameterization to find the arc length from $t = 0$ to $t = 1$.



arc length $\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$

$$= \int_0^1 \sqrt{t^4 + t^2}\, dt = \int_0^1 t\sqrt{t^2+1}\, dt$$

$u = t^2 + 1$

$\frac{du}{dt} = 2t$

$$= \int_1^2 t\, u^{1/2} \frac{dt}{du}\, du = \int_1^2 t u^{1/2} \cdot \frac{1}{2t}\, du = \int_1^2 \frac{1}{2} u^{1/2}\, du$$

$$= \left[\frac{1}{2}\frac{2}{3} u^{3/2}\right]_1^2 = \frac{2\sqrt{2}}{3} - \frac{1}{3}.$$

(6) Find $\int \frac{3}{x^2 - x - 2}\, dx$.

$$\frac{3}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{(x+1)(x-2)}$$

$x=-1$: $3 = -3B$ $\quad B=-1$

$x=2$: $3 = 3A$ $\quad A=1$

$$\int \frac{1}{x-2} - \frac{1}{x+1}\, dx = \ln|x-2| - \ln|x+1| + C$$

(7) Find the degree three Taylor polynomial for $\ln(1+\sqrt{x})$ centered at $x=1$.

$$f(x) = \ln(1+x^{1/2})$$

$$f'(x) = \frac{1}{1+x^{1/2}} \cdot \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}+2x} = \frac{1}{2}(x+x^{1/2})^{-1}$$

$$f''(x) = -\frac{1}{2}(x+x^{1/2})^{-2} \cdot (1+\tfrac{1}{2}x^{-1/2})$$

$$f^{(3)}(x) = 1\cancel{4}(x+x^{1/2})^{-3} \cdot (1+\tfrac{1}{2}x^{-1/2})^2 - \frac{1}{2}(x+x^{1/2})^{-2} \cdot (-\tfrac{1}{4}x^{-3/2}).$$

$$f(1) = \ln(2)$$

$$f'(1) = \cancel{2}\,1/4$$

$$f''(1) = \frac{3/2}{-2\cdot 4} = -\frac{3}{16}$$

$$f'''(1) = \cancel{4}1 \cdot \frac{1}{8} \cdot \frac{9}{4} - \frac{\cancel{8}1}{2} \frac{1}{4} \cdot -\frac{1}{4} = \frac{9}{32\cancel{8}} + \frac{1}{\cancel{8}32} = \cancel{\frac{8}{9}}\frac{10}{32} = \frac{5}{16}$$

$$T_3(x) = \ln(2) + \frac{(x-1)}{4} - \frac{3}{16}\frac{(x-1)^2}{2!} + \cancel{\frac{5}{4}}\frac{(x-1)^3}{3!}\,\cancel{\frac{13}{32}}\,\frac{5}{16}$$

(8) (a) What is a sequence?

A sequenu is an ordered list of numbers indexed by $\mathbb{N}$, e.g. 1, 2, 3, ...

(b) What is a series?

A series is an infinite sum of numbers (indexed by $\mathbb{N}$) e.g. $1 + \frac{1}{2} + \frac{1}{4} + \dots$

(c) If $a_n \to 0$ as $n \to \infty$, does $\sum_{1}^{\infty} a_n$ converge? If so explain why. If not, give an example.

No, $\frac{1}{n} \to 0$ by $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges.

(9) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

ratio test : $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \right|$

$= \lim_{n\to\infty} \frac{1}{2} \frac{n^2+2n+1}{n^2} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) = \frac{1}{2} < 1$

so converges.

(10) Explain whether the following series converges or diverges, indicating clearly which tests you use.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}.$$

Limit comparison test $b_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^3}} = 1$$

$\sum \frac{1}{n^{3/2}}$ converges (p-series w/ $p>1$)

so $\sum \frac{1}{\sqrt{n^3+1}}$ converges.

(11) Find the power series for e^{x^3}, centered at $x=0$. You may use the power series for e^x without justification. Use this to find a power series for $\int e^{x^3}\,dx$. Find the radius of convergence for this power series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \cdots + \frac{x^{3n}}{n!} + \cdots$$

$$\int e^{x^3}\,dx = x + \frac{x^4}{4} + \frac{x^7}{7 \cdot 2!} + \cdots + \frac{x^{3n+1}}{(3n+1)\,n!} + \cdots$$

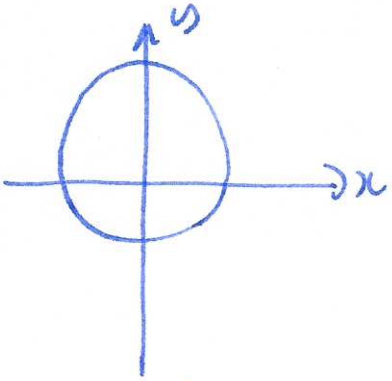
ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{3n+4}}{(3n+4)(n+1)!} \frac{(3n+1)\,n!}{x^{3n+1}} \right|$

$= \lim_{n\to\infty} |x^3| \frac{3n+1}{3n+4} \frac{1}{(n+1)} = 0$ so converges for all x.

$(R = \infty)$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$

(12) Sketch the polar coordinate graph $r = \sin(\theta) + 3$ and find the area bounded by the curve.



$$\text{area} = \int_0^{2\pi} \frac{1}{2}(\sin\theta + 3)^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\sin^2\theta + 3\sin\theta + 9 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} - \frac{1}{4}\cos 2\theta + 3\sin\theta + 9 \, d\theta$$

$$= \left[\frac{1}{4}\theta - \frac{1}{8}\sin 2\theta - 3\cos\theta + 9\theta\right]_0^{2\pi} = \frac{\pi}{2} + 18\pi = \frac{37\pi}{2}$$