Unpin Video Recording...

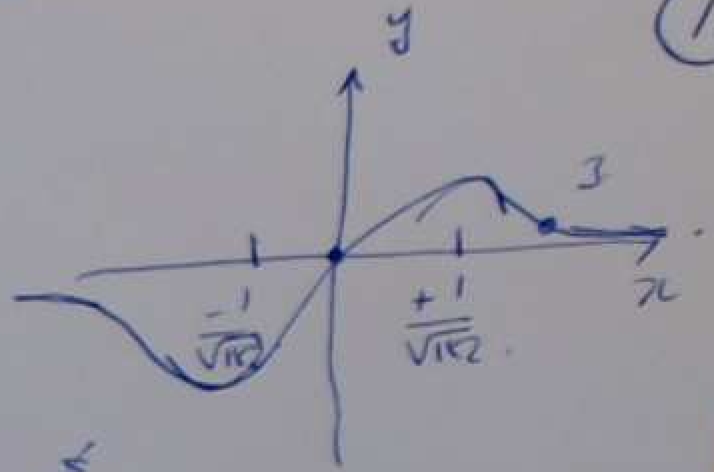
①

HW 4.6 Q4

$y = \underbrace{x}_{a}\underbrace{e^{-91x^2}}_{b}$

$(ab)' = a'b + ab'$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$



$y' = 1e^{-91x^2} + x \cdot e^{-91x^2} \cdot -182x$

$y' = e^{-91x^2}(1 - 182x^2)$ ← first derivative test

$x = \pm \frac{1}{\sqrt{182}}$ ← critical points

$y'' = e^{-91x^2} \cdot -182x\,(1 - \overset{182}{\cancel{872}}x^2) + e^{-91x^2}(-364x)$

$= e^{-91x^2}(-182x + 182x^2 - 364x)$

$= 182\,e^{-91x^2}(x^2 - 3x)$

$x(x-3)$

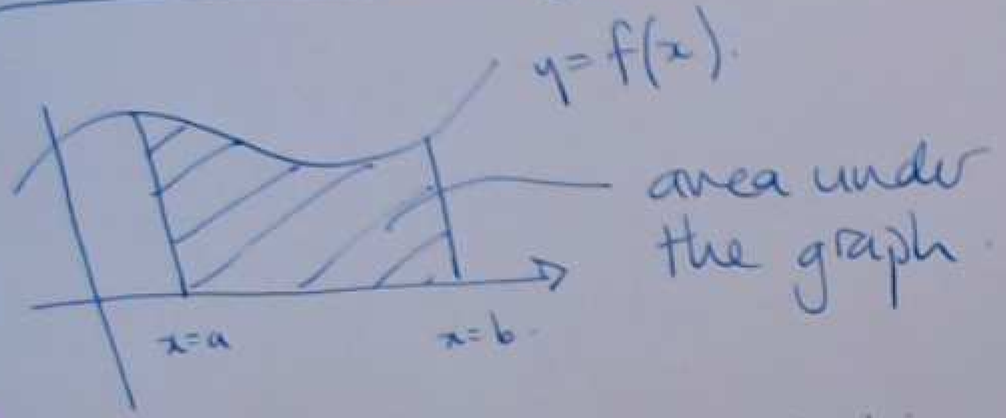
inflection points

$x = 0$

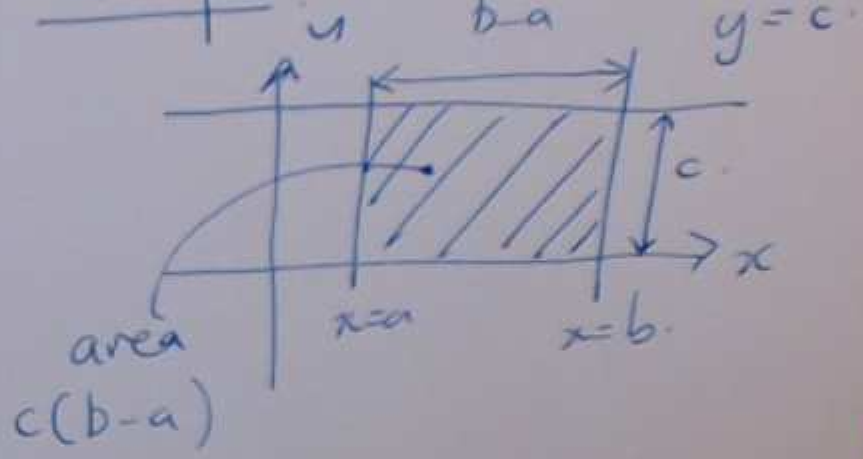
$x = 3$

Joseph Maher

§5.1 Approximating area



Example



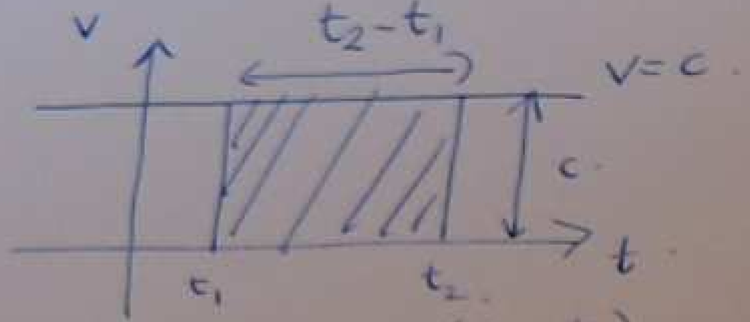
Example plot velocity against time

travel at constant speed $v=c$.

distance travelled

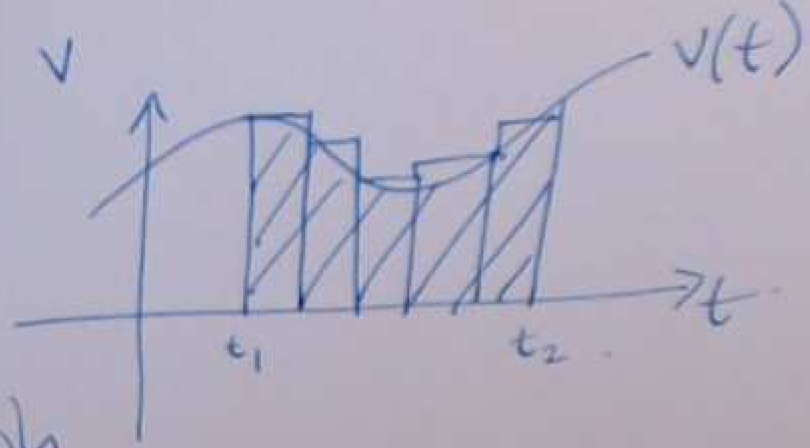
= velocity × time

= area under the graph



distance = $c(t_2-t_1)$

non-constant velocity:

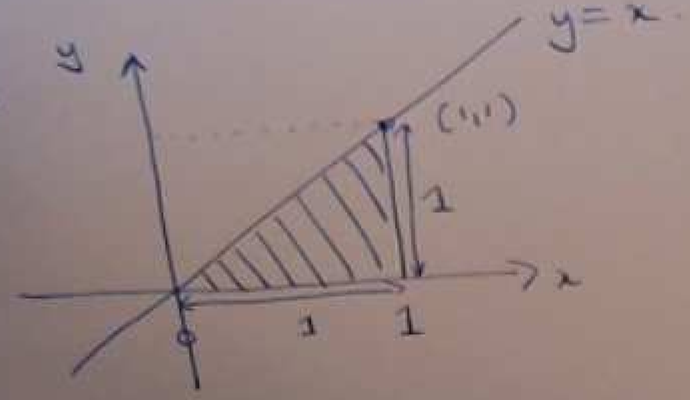


distance travelled

= area under the graph

finding the area: approximate by rectangles.

Example



area under graph

$= $ area of triangle

$= \frac{1}{2}$ base × height

$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

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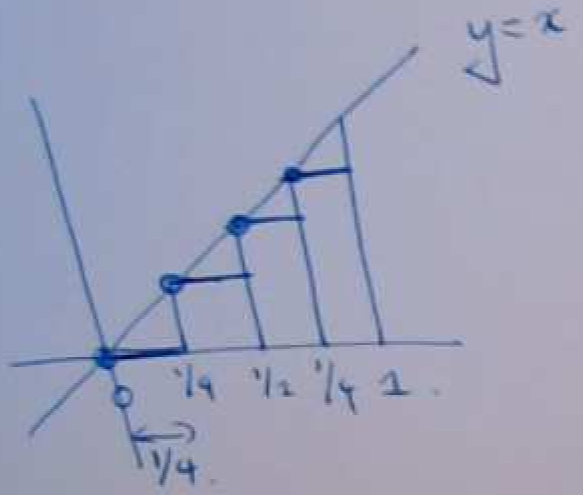
approximate by 4 rectangles.



area $\approx$ sum of areas of the rectangles = width × height

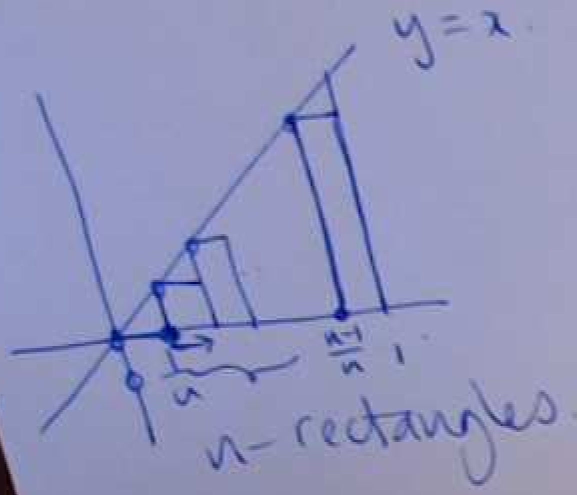
$$= \frac{1}{4} f(0) + \frac{1}{4} f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right)$$

$$= \sum_{i=0}^{3} \frac{1}{4} f\left(\frac{i}{4}\right)$$

$$= \frac{1}{4}\left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{2}{4}\right) + f\left(\frac{3}{4}\right) \right)$$

$$= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{2}{4} + \frac{3}{4} \right)$$

$$= \frac{1}{4} \cdot \frac{1}{4} (0 + 1 + 2 + 3) = \frac{6}{16} = \frac{3}{8} = 0.375$$



approximate with n rectangles.

width = $\cancel{\frac{1}{100}}$ $\frac{1}{n}$

heights $f(0), f(\frac{1}{n}), f(\frac{2}{n}), f(\frac{3}{n}), \ldots$

$$\frac{1}{n} f(0) + \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right)$$

$$\frac{1}{n}\left(f(0) + f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) \right)$$

$$\frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n-1}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n} \cdot \frac{i}{n}$$

$$\frac{1}{n^2}\left(0 + 1 + 2 + 3 + \cdots + n-1\right) = \frac{1}{n^2} \sum_{i=0}^{n-1} i$$

claim: $1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1)$

claim $1+2+3+4\cdots+n = \frac{1}{2}n(n+1)$

1	1
$1+2$	3
$1+2+3$	6

at $k+1$: $\frac{1}{2}(k+1)(k+1+1)$

Proof

① induction: assume true for $n=k$.

$$S_k = 1+2+3+\cdots+k = \frac{1}{2}k(k+1).$$

$$S_{k+1} = \underbrace{1+2+3+\cdots+k}_{S_k}+k+1 = \frac{1}{2}k(k+1)+k+1$$

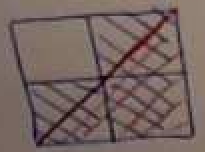
$$= (k+1)\left(\frac{1}{2}k+1\right)$$

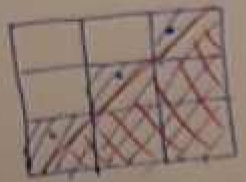
$$\frac{1}{2}(k+1)(k+2) \checkmark$$

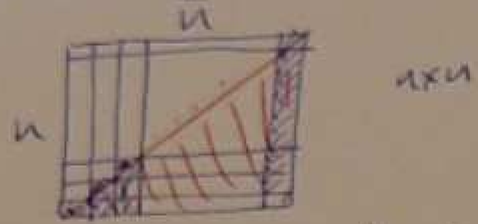
shows true for $k \Rightarrow$ true for $k+1$.

check base case works for $k=1$: $1 = \frac{1}{2}1\times 2 = 1$ ✓ □

② $1+2$







$\frac{1}{2}(2)^2+2\cdot\frac{1}{2}$

$2+1=3$

$\frac{1}{2}3^2+\frac{1}{2}3$

$\frac{1}{2}n^2+\frac{1}{2}n = \frac{1}{2}n(n+1)$

(3) $1+2+3+\cdots+n-1+n$

$n+1$

$n+1$

n even.

$\frac{n}{2}(n+1) = \frac{1}{2}n(n+1)$

n odd

$1+2+3+\cdots \frac{n+1}{2} \cdots n-1+n$

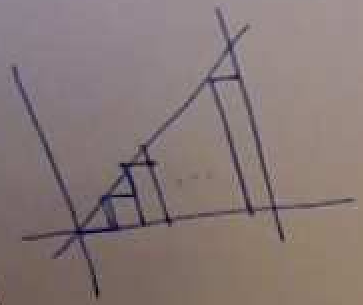
$n+1$

$n+1$

1 2 3.

$\frac{n-1}{2}(n+1) + \frac{n+1}{2} = (n+1)\frac{1}{2}(n-1+1) = \frac{1}{2}n(n+1)$ ✓ □

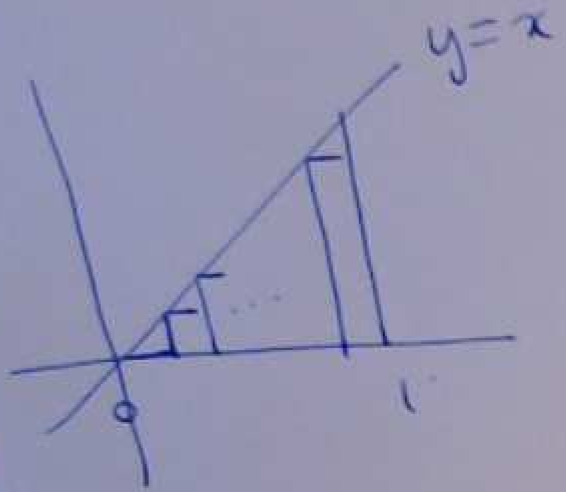
approximate with n-rectangles.



area $\approx \frac{1}{n^2}(0+1+2+\cdots n-1)$

$= \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{n^2-n}{2n^2}$

$= \frac{1}{2} - \frac{1}{2n}$

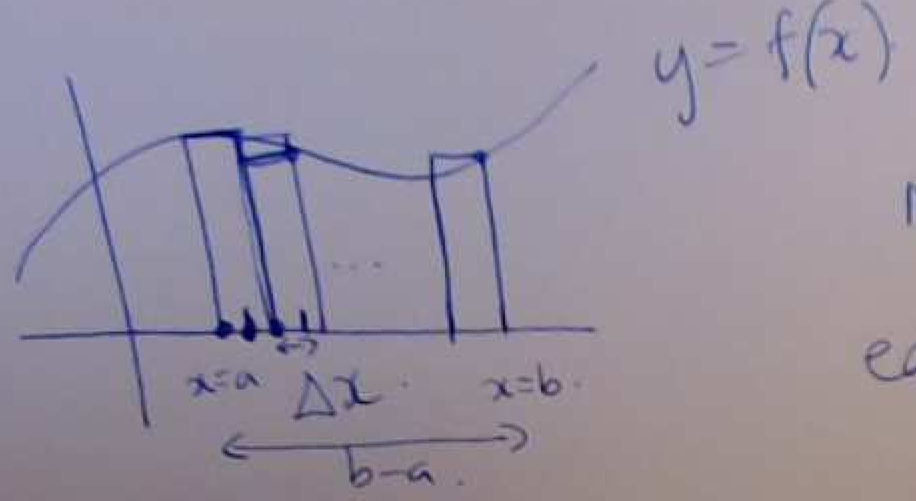


n-rectangles have

$$\text{area} = \frac{n^2 - n}{2n^2} = \frac{1}{2} - \frac{1}{2n}$$

$$\lim_{n \to \infty} \frac{1}{2} - \frac{1}{2n} = \frac{1}{2}$$

Notation



N rectangles

equal width

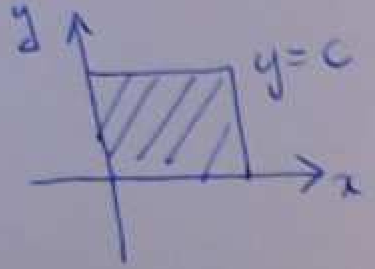
$$\Delta x = \frac{b-a}{N}$$

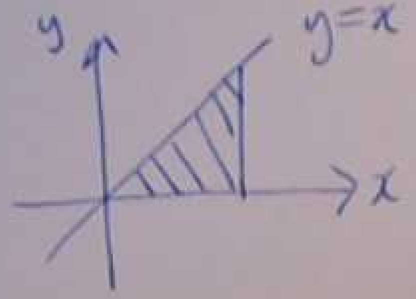
left endpoint rectangles $\quad L_N = \sum_{i=0}^{N-1} f(a + i\Delta x)\,\Delta x$

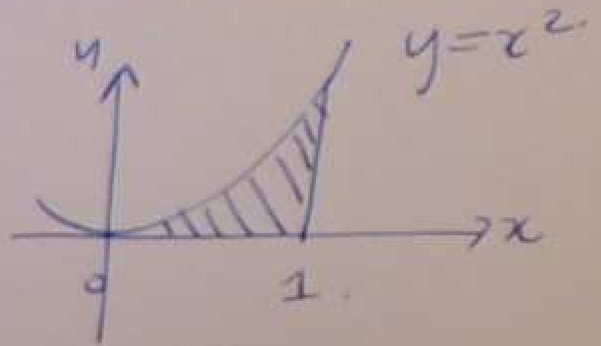
right endpoint rectangles $\quad R_N = \sum_{i=1}^{N} f(a + i\Delta x)\,\Delta x$

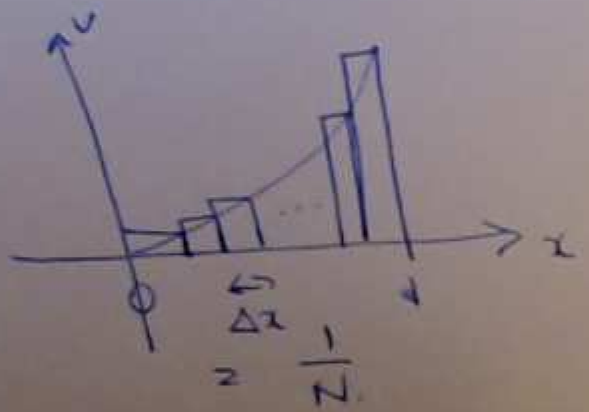
midpoints rectangles $\quad M_N = \sum_{i=1}^{N} f\left(a + \left(i - \tfrac{1}{2}\right)\Delta x\right)\Delta x$

Examples









$$R_N = \sum_{i=1}^{N} \frac{1}{N} f\left(\frac{i}{N}\right)$$

$$= \sum_{i=1}^{N} \frac{1}{N} \cdot \frac{i^2}{N^2} = \frac{1}{N^3} \sum_{i=1}^{N} i^2$$

$$= \frac{1}{N^3}\left(1^2 + 2^2 + 3^2 + \cdots + N^2\right)$$

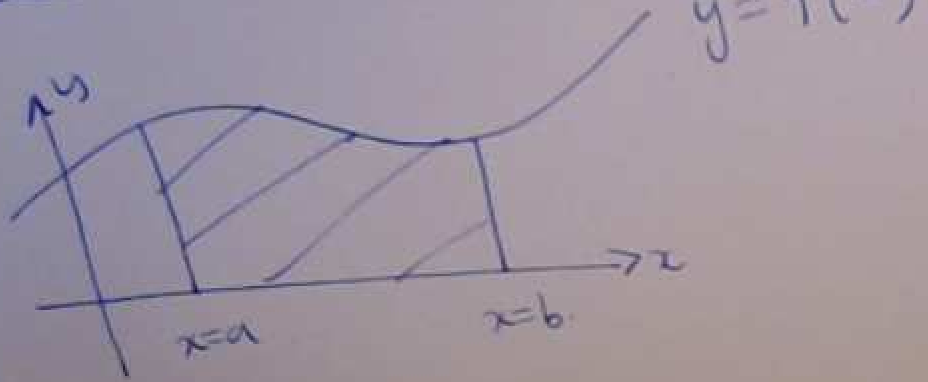
Fact $\frac{1}{6} N(N+1)(2N+1)$

useful fact

<u>Thm</u> If $f(x)$ is continuous on $[a,b]$ then all of these approximations have the same limit as $N \to \infty$, which is equal to the area under the curve

$$\text{area} = \lim_{N\to\infty} L_N = \lim_{N\to\infty} R_N = \lim_{N\to\infty} M_N.$$

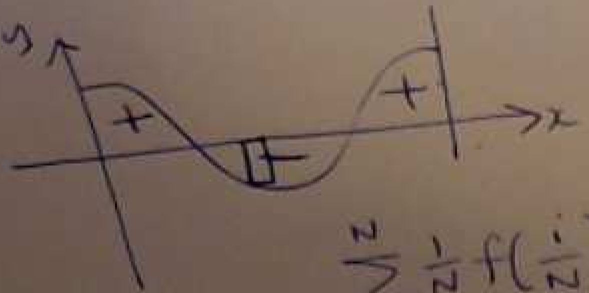
§ 5.2 Definite integrals



$$\int_a^b f(x)\,dx$$

= area under the curve from $x=a$ to $x=b$.

note: signed area.



$$\sum_{i=1}^{N} \frac{1}{N} f\left(\frac{i}{N}\right)$$

so $\int_0^{2\pi} \cos(x)\,dx = 0$.

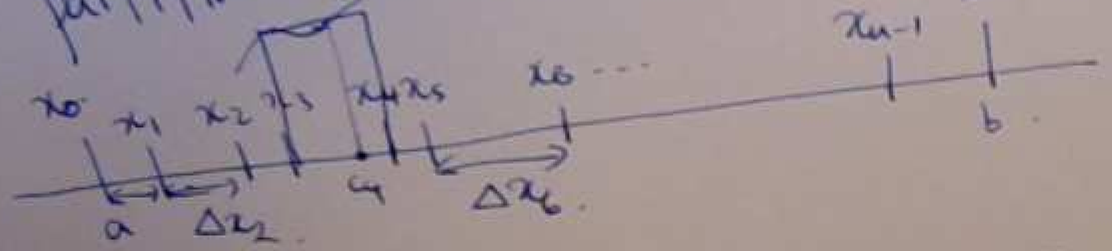
$\int_0^{2\pi} \sin(x)\,dx = 0$.

Formal definition

Riemann sum $R(f, P, C)$

function

P partition of $[a, b]$



widths

$\Delta x_i = x_i - x_{i-1}$

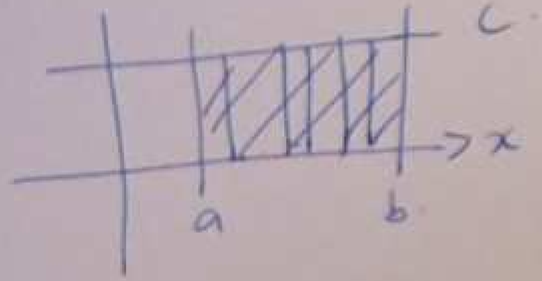
C = choice of a point $c_i \in [x_{i-1}, x_i]$

$$R(f, P, C) = \sum_{i=1}^{n} f(c_i)\Delta x_i \qquad \|P\| = \max \Delta x_i$$

<u>Defn</u> $\displaystyle\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} R(f, P, C)$

when this limit exists, we say f is integrable on $[a, b]$

useful properties



$$\int_a^b c\,dx = c(b-a).$$

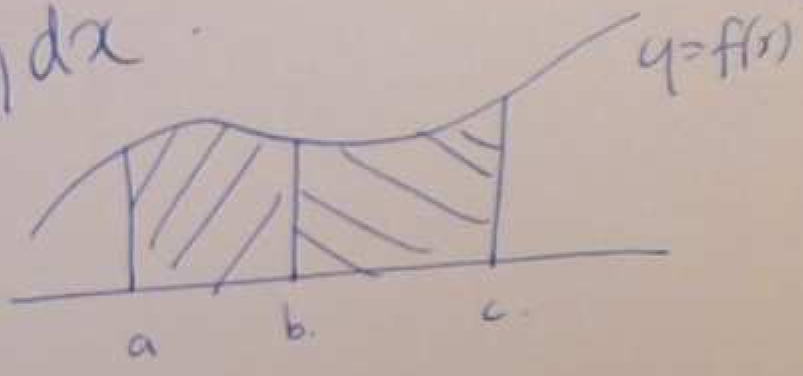
$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

0-length interval $\int_a^a f(x)\,dx = 0$

adjacent intervals.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$



reversing limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

examples

$$\int_a^b f(x) + \int_b^a f(x)\,dx = \underbrace{\int_a^a f(x)\,dx}_{\text{0-length interval}} = 0.$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

comparison if $f(x) \le g(x)$.

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

§5.3 Antiderivative

Def^n^ A function $F(x)$ is an antiderivative of $f(x)$ is $F'(x) = f(x)$.

Example if $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an antiderivative for x^2.

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3}\cdot 3x^2 = x^2 \quad \checkmark$$

note $F(x) = \frac{1}{3}x^3 + 7$ is also an anti-derivative for x^2

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + 7\right) = \frac{1}{3}\cdot 3x^2 = x^2 \quad \checkmark$$

General antiderivative

Thm Let $F(x)$ be an antiderivative for $f(x)$, then any other antiderivative has the form $F(x) + c$ for some constant $c \in \mathbb{R}$

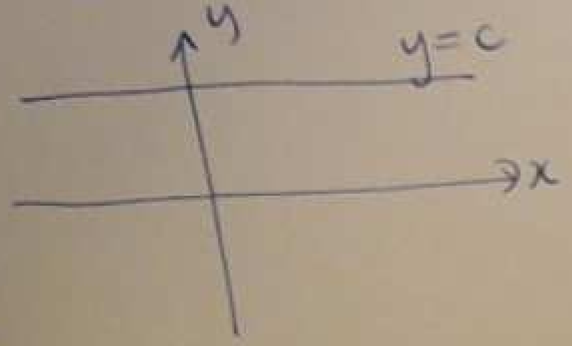
Proof Let $F(x), G(x)$ be antiderivatives for $f(x)$ then

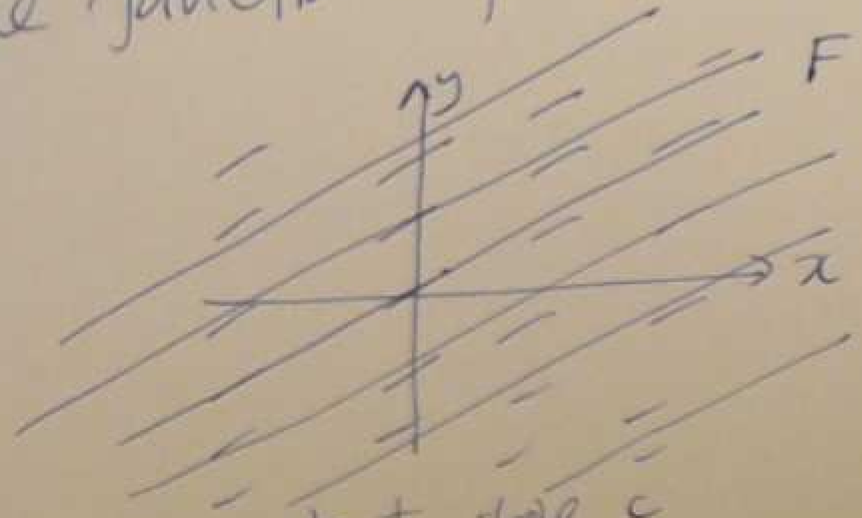
$(F(x) - G(x))' = F'(x) - G'(x) = f(x) - f(x) = 0 \Rightarrow F(x) - G(x) = \text{const}$ $\square$

Picture $f(x)$ gives the slope function for $F(x)$. (18)

Example

$f(x) = c$

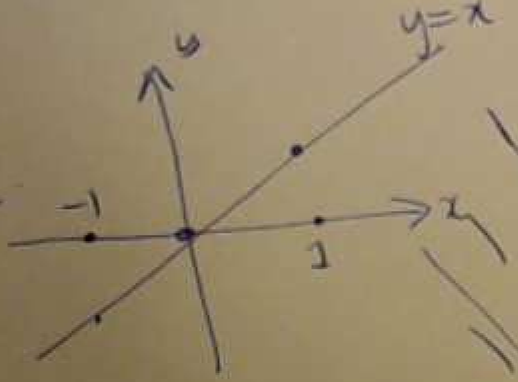


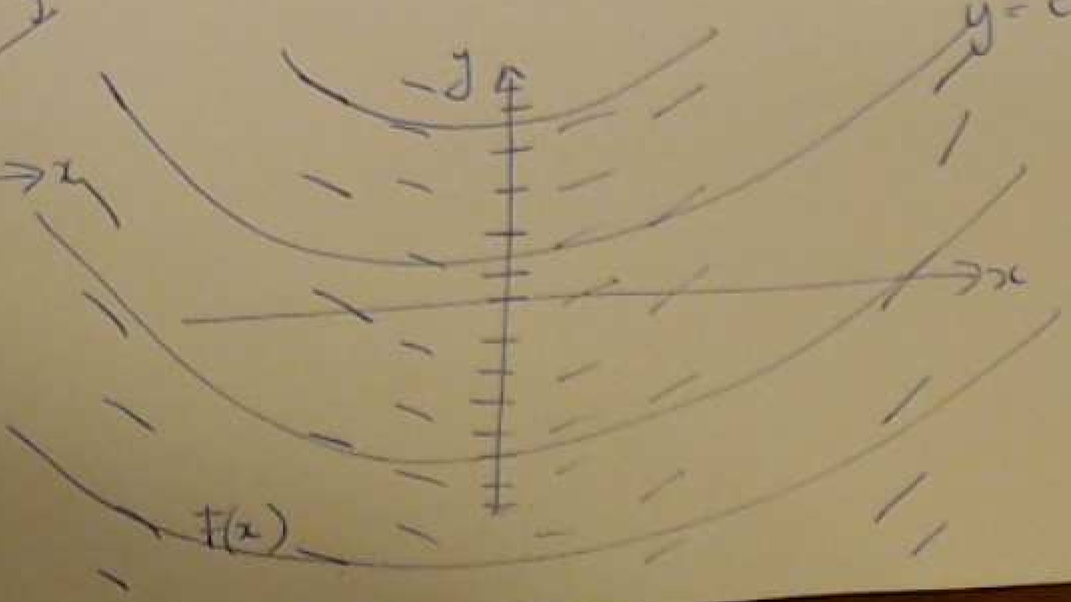


constant slope c

$y = cx + d$.

$f(x) = x$





$F(x) = \frac{1}{2}x^2 + c$.